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**CONTINUOUS AND DEGENERATE
GRAVITATING SYSTEMS**

BY

R. G. LANGEBARTEL

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Code 1



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SUMMARY

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The methods of statistical mechanics used in stellar dynamics are combined with the methods of Schwartz distribution theory to treat various types of continuous and discontinuous media. The motion is considered in phase space. One example handled is that of a medium continuous in position space but discrete in velocity space. Another is that of a medium discrete in both position and velocity space, i.e. the n-body problem. Thus, this presents for celestial mechanics an alternative method to the classical one.

Introduction:

The usual practice in most problems in celestial mechanics is to set up the equations of motion for the particle or particles and then effect their integration by one means or another. The purpose of this report is to examine some of these problems from the point of view of statistical mechanics. This branch of mechanics deals primarily with the properties of the density distribution function for the medium in phase space. In general this is regarded as a continuous function in both position and velocity variables, thus describing a continuous medium. One can, however, regard it as representing, say, a number of discrete particles by expressing the density distribution function in terms of the "singularity functions"--the Dirac delta functions and their derivatives. This approach to the n-body problem is equivalent to the standard one but it presents the equations in a novel form which may serve to suggest new methods of attack.

In the case of a very large number of particles a convenient procedure when applicable is to regard the density function as continuous in the spatial variables but discrete in the velocity variables. Such a medium lies midway between a completely continuous medium and a collection of n bodies. It is in general easier to handle than the former while still maintaining in many instances the properties of interest of the former.

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The following table shows the results of the survey conducted in the year 1991. The table is divided into two main sections: 'General Information' and 'Detailed Information'. The 'General Information' section includes the total number of respondents, the number of respondents who are currently employed, and the number of respondents who are currently unemployed. The 'Detailed Information' section includes the number of respondents who are currently employed in each of the following categories: 'Manufacturing', 'Services', 'Government', and 'Other'. The table also includes the number of respondents who are currently unemployed in each of the following categories: 'Unemployed', 'Retired', 'Disabled', and 'Other'.

I. The Fundamental Equation:

The density distribution function is subject to the principal condition that it satisfy Liouville's Equation, also known as the Equation of Continuity. This is a statement to the effect that the density of the medium throughout its motion remains constant in phase space. The forces are assumed to be conservative with the force vector taken as the negative of the gradient of the potential.

Let q_r, p_r be the n Hamiltonian coordinates and their conjugate momenta, and $H(q_r, p_r, t)$ the Hamiltonian function. "Phase space" is the $2n$ -dimensional space of points $(q_1, \dots, q_n, p_1, \dots, p_n)$. Let $f(q_r, p_r, t)$ represent the density distribution function in phase space. Then Liouville's Theorem (1, p. 83; 4, p. 266) asserts that

$$\frac{\partial f}{\partial t} + \sum_k \left(\frac{\partial H}{\partial p_k} \frac{\partial f}{\partial q_k} - \frac{\partial H}{\partial q_k} \frac{\partial f}{\partial p_k} \right) = 0. \quad (1)$$

This first order partial differential equation in f is the fundamental equation for this theory. If the forces involved are purely external then this equation is linear. But if the self-gravitation of the particles is taken into account then f enters into H through the potential function V , and the equation becomes a non-linear integro-differential equation which must in almost every case be handled by some approximation procedure.

Liouville's Equation (1) can be derived directly from the Hamiltonian Equation of Motion but it is instructive for the case of a degenerate medium to show the equivalence of (1) with the Newtonian Theory by making use of the delta function. Such "functions" have been employed in applications for a number of years but it is only relatively recently that a satisfactory theory incorporating them has been evolved (9, 6, 8). In particular, the Dirac

delta function, $\delta(x)$, is to be such that for every continuous function

$$\int_{-\infty}^{\infty} \delta(x) \varphi(x) dx = \varphi(0). \quad (2)$$

But it is not difficult to show that this definition is in violation of any standard integration theory (i.e. Riemann, Lebesgue, etc.) inasmuch as it implies that $\delta(x)$ is zero everywhere except at the origin, and for such an integrand the integral must be zero. Modern developments in the theory of these "weak functions" have succeeded in generalizing the notion of the integral adequately to cover formulae such as (2) (cf. ref. 8). The physical interpretation of $\delta(x)$ is that it represents the density distribution function of a unit mass concentrated at one point, the origin.

Let us suppose that the motion of a certain system with n degrees of freedom is given by

$$\begin{cases} q_n = Q_n(q_s, p_s, t), \\ p_n = P_n(q_s, p_s, t). \end{cases} \quad (3)$$

Then the phase space density function for such a system would be given by

$$f = \delta(q_n - Q_n, p_n - P_n) \equiv \delta(q_1 - Q_1, q_2 - Q_2, \dots, p_n - P_n) \quad (4)$$

and we are to show that substitution of (4) into (1) leads to a result equivalent to that predicted by the Newtonian Theory. With the notation

$$\begin{cases} \delta_n = \frac{\partial \delta(q_s - Q_s, p_s - P_s)}{\partial (q_n - Q_n)}, \\ \delta_{n+n} = \frac{\partial \delta(q_s - Q_s, p_s - P_s)}{\partial (p_n - P_n)}, \end{cases}$$

we find that (1) under the substitution (4) takes on the form

$$\begin{aligned} & \left\{ \sum_k \left[\sum_n \left(H_{P_k P_n} \frac{\partial P_n}{\partial q_k} + H_{P_k Q_n} \frac{\partial Q_n}{\partial q_k} \right) - \sum_n \left(H_{Q_k P_n} \frac{\partial P_n}{\partial p_k} + H_{Q_k Q_n} \frac{\partial Q_n}{\partial p_k} \right) \right] \right\} \delta \\ & - \sum_n \left[\frac{\partial Q_n}{\partial t} + \sum_k \left(H_{P_k} \frac{\partial Q_n}{\partial q_k} - H_{Q_k} \frac{\partial Q_n}{\partial p_k} \right) - H_{P_n} \right] \delta_n \\ & - \sum_n \left[\frac{\partial P_n}{\partial t} + \sum_k \left(H_{P_k} \frac{\partial P_n}{\partial q_k} - H_{Q_k} \frac{\partial P_n}{\partial p_k} \right) + H_{Q_n} \right] \delta_{n+n} = 0, \end{aligned} \quad (5)$$

where $H_{P_k P_n} \equiv \frac{\partial^2 H(Q_s, P_s, t)}{\partial P_n \partial P_k}$; i.e. the original variables q_r, p_r in H are replaced by the functions Q_r, P_r . In deriving (5) use is made of formulae for the multiplication of the delta function or its derivative by another function. Particular formulae employed in this instance are ((6, p. 10))

$$\begin{cases} H_{q_k}(q_s, p_s, t) \delta_n = H_{q_k}(Q_s, P_s, t) \delta_n - H_{q_k p_n}(q_s, p_s, t) \delta = H_{Q_k} \delta_n - H_{Q_k Q_n} \delta, \\ H_{p_k}(q_s, p_s, t) \delta_{n+n} = H_{p_k}(Q_s, P_s, t) \delta_{n+n} - H_{p_k p_n}(q_s, p_s, t) \delta = H_{Q_k} \delta_{n+n} - H_{Q_k P_n} \delta. \end{cases}$$

Setting the coefficients of δ_r and δ_{n+r} to zero gives a partial differential equation system in Q_r, P_r of a rather special form. It is known as a "quasi-linear system with same principal part" ([2, vol. II, p. 117]) and its solution can be expressed as an arbitrary function of $2n$ certain functions of q_r, p_r, t . These functions in the case we are dealing with turn out to be precisely the Newtonian integrals of motion for the given Hamiltonian. Observe that the coefficient of δ in (5) can be written in the form

$$\sum_k \left(\frac{\partial H_k}{\partial q_k} - \frac{\partial H_{q_k}}{\partial p_k} \right),$$

and by virtue of the property $g(x)\delta(x-a) = g(a)\delta(x-a)$ it follows that the δ term is

$$\begin{aligned} & \sum_k \left(\frac{\partial H_k(q_n, p_n, t)}{\partial q_k} - \frac{\partial H_{q_k}(q_n, p_n, t)}{\partial p_k} \right) \delta(q_n - Q_n, p_n - P_n) \\ &= \sum_k (H_{q_k} - H_{p_k}) \delta = 0. \end{aligned}$$

That (5) does represent the Newtonian theory can be verified in a much briefer fashion by considering the special case $Q_r = Q_r(t)$, $P_r = P_r(t)$, the form in which the solution to a problem in dynamics is most often sought. For Q_r and P_r independent of q_r, p_r the coefficients of δ_r and δ_{n+r} reduce merely to

$$\begin{cases} \frac{dQ_r}{dt} = H_{p_r}, \\ \frac{dP_r}{dt} = -H_{q_r}, \end{cases}$$

i.e. P_r and Q_r satisfy the Hamiltonian Equations of Motion. Note also that

for P_r and Q_r independent of q_r, p_r each term in the coefficient of δ in (5) vanishes.

Other choices as to what variables are to be considered present in Q_r and P_r lead to certain specialized forms for the quasi-linear system derived from the coefficients of δ_r and δ_{n+r} in (5). This permits a freedom in the choice of equations to use for a particular problem.

Example: Harmonic Oscillator. The Hamiltonian is $H = \frac{1}{2}(q^2 + p^2)$ and let us take $Q = Q(t)$, $P = P(q, t)$. From (5) we obtain the equation for $P(q, t)$:

$$-\frac{\partial P}{\partial t} - P \frac{\partial P}{\partial q} - q = 0.$$

The fact that H is independent of explicit t in this example prompts us to assume this is also the case for P . Under this assumption the partial differential equation for P becomes a total differential equation,

$$P \frac{dP}{dq} + q = 0,$$

for which the solution is easily found to be

$$P = \pm \sqrt{c_1^2 - q^2}.$$

The remaining equation taken from (5) is then merely

$$\frac{dQ}{dt} - \sqrt{c_1^2 - Q^2} = 0,$$

so that the complete solution is

$$\begin{cases} Q = c_1 \sin(t - c_2), \\ P = \pm \sqrt{c_1^2 - c_1^2 \sin^2(t - c_2)} = c_1 \cos(t - c_2). \end{cases}$$

It might be that the equations taken from (5) with the assumption $P_r = P_r(t)$, $Q_r = Q_r(p_s, t)$ are easier to handle than those resulting from the assumption $P_r = P_r(q_s, t)$, $Q_r = Q_r(t)$ for some particular problem. Employing the former assumption rather than the latter (a sort of a dual situation) is somewhat analogous to the use of the Legendre transformation in partial differential equations ((2, vol. II, p. 26)).

II. The n-body Problem:

Let z_1, z_2, z_3 be rectangular coordinates and let Z_1, Z_2, Z_3 be the corresponding velocity components. Let $\frac{\partial U}{\partial \dot{z}_n}$ be the negative of the force per unit mass due to the self-gravitation of the particles themselves and let $F_r(z_s, t)$ be the external force per unit mass (due for example to the mass of a body not taken as a part of the n-body system and regarded as not affected by the masses of the n-bodies). Then Liouville's Equation (1) in these rectangular coordinates is

$$\frac{\partial f}{\partial t} + \sum_{n=1}^3 \dot{Z}_n \frac{\partial f}{\partial z_n} - \sum_{n=1}^3 \left(\frac{\partial U}{\partial \dot{z}_n} - F_n \right) \frac{\partial f}{\partial \dot{Z}_n} = 0. \quad (6)$$

The density distribution function that describes the n-body system is given by

$$f = \sum_{\alpha=1}^n m_{\alpha} \delta(z_n - \tilde{z}_n^{[\alpha]}, \dot{Z}_n - \tilde{\dot{Z}}_n^{[\alpha]}) \equiv \sum_{\alpha=1}^n m_{\alpha} \delta^{[\alpha]} \quad (7)$$

In formula (7) $\tilde{x}_n^{[a]}$, $\tilde{z}_n^{[a]}$ are the position and velocity coordinates of the α^{th} particle, and z_r , z_r are the running position and velocity coordinates throughout the space. We shall work out the form that Liouville's Equation takes under substitution of (7) with the assumption that $\tilde{x}_n^{[a]}$, $\tilde{z}_n^{[a]}$ are functions of all the variables z_r, z_r, t although in practice they would most likely be considerably restricted in this regard--quite often to being simply functions of t alone. A case of considerable importance (already mentioned in the previous section) is that of taking $\tilde{x}_n^{[a]}$ as function of t alone but $\tilde{z}_n^{[a]}$ as function of both z_s and t . Examples of this are given shortly.

To obtain the density distribution, N , in position space--the same kind of density as found for example in hydrodynamics--one integrates the phase space density function, f , over all velocity space:

$$N = \iiint_{-\infty}^{\infty} f d\tilde{z}_1 d\tilde{z}_2 d\tilde{z}_3.$$

Consequently, the potential U due to the n -bodies is

$$\begin{aligned} U &= -G \iiint_{-\infty}^{\infty} \frac{N(\tilde{x}_n, t) d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_3}{\sqrt{(\tilde{x}_1 - \tilde{x}_1)^2 + (\tilde{x}_2 - \tilde{x}_2)^2 + (\tilde{x}_3 - \tilde{x}_3)^2}}, \\ &= -G \iiint_{-\infty}^{\infty} \iiint_{-\infty}^{\infty} \frac{\sum_{\alpha=1}^n m_{\alpha} \delta(\tilde{x}_n - \tilde{x}_n^{[a]}, \tilde{z}_n - \tilde{z}_n^{[a]}) d\tilde{z}_1 d\tilde{z}_2 d\tilde{z}_3 d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_3}{\sqrt{\sum_s (\tilde{x}_s - \tilde{x}_s)^2}}, \\ &= -G \iiint_{-\infty}^{\infty} \frac{\sum_{\alpha=1}^n m_{\alpha} \delta(\tilde{x}_n - \tilde{x}_n^{[a]}) d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_3}{\sqrt{\sum_s (\tilde{x}_s - \tilde{x}_s)^2}}, \\ &= -G \sum_{\alpha=1}^n \frac{m_{\alpha}}{\sqrt{\sum_s (\tilde{x}_s - \tilde{x}_s^{[a]})^2}}. \end{aligned}$$

It should be realized that in this final expression for U the functions $\tilde{z}_s^{[a]}$ no longer involve z_r and $\tilde{z}_r$ (if they ever did) since these have been integrated out.

We again employ the notation

$$\left\{ \begin{array}{l} \delta_s^{[a]} \equiv \frac{\partial \delta(\tilde{z}_n - \tilde{z}_n^{[a]}, \tilde{z}_n - \tilde{z}_n^{[a]})}{\partial (\tilde{z}_s - \tilde{z}_s^{[a]})}, \\ \delta_{3+s}^{[a]} \equiv \frac{\partial \delta(\tilde{z}_n - \tilde{z}_n^{[a]}, \tilde{z}_n - \tilde{z}_n^{[a]})}{\partial (\tilde{z}_s - \tilde{z}_s^{[a]})}, \end{array} \right.$$

which simplifies the writing in the expressions that we need:

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial t} = - \sum_{\alpha=1}^n \sum_{s=1}^3 m_{\alpha} \frac{\partial \tilde{z}_s^{[a]}}{\partial t} \delta_s^{[a]} - \sum_{\alpha=1}^n \sum_{s=1}^3 m_{\alpha} \frac{\partial \tilde{z}_s^{[a]}}{\partial t} \delta_{3+s}^{[a]}, \\ \frac{\partial f}{\partial \tilde{z}_n} = \sum_{\alpha=1}^n m_{\alpha} \delta_n^{[a]} - \sum_{\alpha=1}^n \sum_{s=1}^3 m_{\alpha} \frac{\partial \tilde{z}_s^{[a]}}{\partial \tilde{z}_n} \delta_s^{[a]} - \sum_{\alpha=1}^n \sum_{s=1}^3 m_{\alpha} \frac{\partial \tilde{z}_s^{[a]}}{\partial \tilde{z}_n} \delta_{3+s}^{[a]}, \\ \frac{\partial f}{\partial \tilde{z}_n} = \sum_{\alpha=1}^n m_{\alpha} \delta_{3+n}^{[a]} - \sum_{\alpha=1}^n \sum_{s=1}^3 m_{\alpha} \frac{\partial \tilde{z}_s^{[a]}}{\partial \tilde{z}_n} \delta_s^{[a]} - \sum_{\alpha=1}^n \sum_{s=1}^3 m_{\alpha} \frac{\partial \tilde{z}_s^{[a]}}{\partial \tilde{z}_n} \delta_{3+s}^{[a]}. \end{array} \right.$$

Special attention must be paid to the term $\frac{\partial U}{\partial \tilde{z}_n} \delta_s^{[a]}$ since the delta function has the property of changing z_s into $\tilde{z}_s^{[a]}$ and $\frac{\partial U}{\partial \tilde{z}_n}$ is not well defined for $z_s = \tilde{z}_s^{[a]}$. Actually the properties of the delta function are such that the particular terms in $\frac{\partial U}{\partial \tilde{z}_n} \delta_s^{[a]}$ where this difficulty arises are

to be taken to be zero. The discontinuity in $\frac{\partial U}{\partial \tilde{x}_n}$ is such that the limiting value of this function at the point $(\tilde{x}_1^{[\alpha]}, \tilde{x}_2^{[\alpha]}, \tilde{x}_3^{[\alpha]})$ in the troublesome term depends on the direction of approach. For one set of directions the limiting value fails to be well-defined, but for the remaining directions it is well-defined and turns out to be zero. Indeed, if the factor $\delta(\tilde{x}_1 - \tilde{x}_1^{[\alpha]})$ is left to the last in the consideration of the product then we would get

$$\left(\frac{(\tilde{x}_1 - \tilde{x}_1^{[\alpha]}) \delta(\tilde{x}_2 - \tilde{x}_2^{[\alpha]}) \delta(\tilde{x}_3 - \tilde{x}_3^{[\alpha]})}{[(\tilde{x}_1 - \tilde{x}_1^{[\alpha]})^2 + (\tilde{x}_2 - \tilde{x}_2^{[\alpha]})^2 + (\tilde{x}_3 - \tilde{x}_3^{[\alpha]})^2]^{3/2}} \right) \delta(\tilde{x}_1 - \tilde{x}_1^{[\alpha]}) = \frac{\tilde{x}_1 - \tilde{x}_1^{[\alpha]}}{|\tilde{x}_1 - \tilde{x}_1^{[\alpha]}|^3} \delta(\tilde{x}_2 - \tilde{x}_2^{[\alpha]}) \delta(\tilde{x}_3 - \tilde{x}_3^{[\alpha]}) \delta(\tilde{x}_1 - \tilde{x}_1^{[\alpha]})$$

with the meaning not clear. However, if we take it in the other order a definite value is obtained:

$$\begin{aligned} & \left(\frac{(\tilde{x}_1 - \tilde{x}_1^{[\alpha]}) \delta(\tilde{x}_1 - \tilde{x}_1^{[\alpha]})}{[(\tilde{x}_1 - \tilde{x}_1^{[\alpha]})^2 + (\tilde{x}_2 - \tilde{x}_2^{[\alpha]})^2 + (\tilde{x}_3 - \tilde{x}_3^{[\alpha]})^2]^{3/2}} \right) \delta(\tilde{x}_2 - \tilde{x}_2^{[\alpha]}) \delta(\tilde{x}_3 - \tilde{x}_3^{[\alpha]}) \\ &= \left(\frac{0}{[(\tilde{x}_2 - \tilde{x}_2^{[\alpha]})^2 + (\tilde{x}_3 - \tilde{x}_3^{[\alpha]})^2]^{3/2}} \delta(\tilde{x}_1 - \tilde{x}_1^{[\alpha]}) \right) \delta(\tilde{x}_2 - \tilde{x}_2^{[\alpha]}) \delta(\tilde{x}_3 - \tilde{x}_3^{[\alpha]}) = 0. \end{aligned}$$

Moreover, the term physically represents the action of the self-gravitation force of the particle on itself and thus plays no role in determining the motion of the system. Consequently, wherever it occurs we regard this term as zero and on this basis we find

$$\begin{aligned} \frac{\partial U}{\partial \tilde{x}_n} \delta_s^{[\alpha]} &= G \sum_{\beta=1}^n m_\beta \frac{\tilde{x}_n - \tilde{x}_n^{[\beta]}}{[\sum_k (\tilde{x}_k - \tilde{x}_k^{[\beta]})^2]^{3/2}} \delta_s(\tilde{x}_i - \tilde{x}_i^{[\alpha]}, \tilde{x}_i - \tilde{x}_i^{[\alpha]}) \\ &= G \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^n m_\beta \frac{\tilde{x}_n^{[\alpha]} - \tilde{x}_n^{[\beta]}}{[\sum_k (\tilde{x}_k^{[\alpha]} - \tilde{x}_k^{[\beta]})^2]^{3/2}} \delta_s^{[\alpha]} \\ &\quad - G \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^n m_\beta \left\{ - \frac{3(\tilde{x}_n^{[\alpha]} - \tilde{x}_n^{[\beta]})(\tilde{x}_s^{[\alpha]} - \tilde{x}_s^{[\beta]})}{[\sum_k (\tilde{x}_k^{[\alpha]} - \tilde{x}_k^{[\beta]})^2]^{5/2}} + \frac{\delta_n^s}{[\sum_k (\tilde{x}_k^{[\alpha]} - \tilde{x}_k^{[\beta]})^2]^{3/2}} \right\} \delta_s^{[\alpha]}. \end{aligned}$$

In these sums, as it is indicated, the term where $\beta = \alpha$ is to be deleted. This is a manifestation of the vanishing of the weak functions treated above. In the very last term the symbol δ_n^s stands for the Kronecker delta (not the Dirac delta function) and it has the value one for $r = s$ and the value zero for $r \neq s$.

On the basis of these results we then obtain Liouville's Equation in rectangular coordinates for an n-body system to be

$$\begin{aligned}
 & \left[\sum_{\alpha=1}^n \sum_{s=1}^3 m_{\alpha} \frac{\partial \tilde{x}_s^{[\alpha]}}{\partial \tilde{x}_s} + \sum_{\alpha=1}^n \sum_{\beta=1}^n \sum_{s=1}^3 m_{\alpha} \left(-G \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^n m_{\beta} \left\{ -\frac{3(\tilde{x}_n^{[\alpha]} - \tilde{x}_n^{[\beta]})(\tilde{x}_s^{[\alpha]} - \tilde{x}_s^{[\beta]})}{\left[\sum_k (\tilde{x}_k^{[\alpha]} - \tilde{x}_k^{[\beta]})^2 \right]^{3/2}} + \frac{\delta_n^s}{\left[\sum_k (\tilde{x}_k^{[\alpha]} - \tilde{x}_k^{[\beta]})^2 \right]^{3/2}} \right\} \right. \right. \\
 & \quad \left. \left. + \frac{\partial F_n(\tilde{x}_n^{[\alpha]}, t)}{\partial \tilde{x}_s} \right) \frac{\partial \tilde{x}_s^{[\alpha]}}{\partial \tilde{x}_n} \right] \delta_s^{[\alpha]} \\
 & + \sum_{\alpha=1}^n \sum_{s=1}^3 \left[-m_{\alpha} \frac{\partial \tilde{x}_s^{[\alpha]}}{\partial t} - \sum_{n=1}^3 m_{\alpha} \tilde{x}_n^{[\alpha]} \frac{\partial \tilde{x}_s^{[\alpha]}}{\partial \tilde{x}_n} + m_{\alpha} \tilde{x}_s^{[\alpha]} \right. \\
 & \quad \left. + \sum_{n=1}^3 m_{\alpha} \left(G \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^n m_{\beta} \frac{\tilde{x}_n^{[\alpha]} - \tilde{x}_n^{[\beta]}}{\left[\sum_k (\tilde{x}_k^{[\alpha]} - \tilde{x}_k^{[\beta]})^2 \right]^{3/2}} - F_n(\tilde{x}_n^{[\alpha]}, t) \right) \frac{\partial \tilde{x}_s^{[\alpha]}}{\partial \tilde{x}_n} \right] \delta_s^{[\alpha]} \quad (8) \\
 & + \sum_{\alpha=1}^n \sum_{s=1}^3 \left[-m_{\alpha} \frac{\partial \tilde{x}_s^{[\alpha]}}{\partial t} - \sum_{n=1}^3 m_{\alpha} \tilde{x}_n^{[\alpha]} \frac{\partial \tilde{x}_s^{[\alpha]}}{\partial \tilde{x}_n} \right. \\
 & \quad \left. + \sum_{n=1}^3 m_{\alpha} \left(G \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^n m_{\beta} \frac{\tilde{x}_n^{[\alpha]} - \tilde{x}_n^{[\beta]}}{\left[\sum_k (\tilde{x}_k^{[\alpha]} - \tilde{x}_k^{[\beta]})^2 \right]^{3/2}} - F_n(\tilde{x}_n^{[\alpha]}, t) \right) \frac{\partial \tilde{x}_s^{[\alpha]}}{\partial \tilde{x}_n} \right. \\
 & \quad \left. - m_{\alpha} \left(G \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^n m_{\beta} \frac{\tilde{x}_s^{[\alpha]} - \tilde{x}_s^{[\beta]}}{\left[\sum_k (\tilde{x}_k^{[\alpha]} - \tilde{x}_k^{[\beta]})^2 \right]^{3/2}} - F_s(\tilde{x}_s^{[\alpha]}, t) \right) \right] \delta_{s+s}^{[\alpha]} = 0.
 \end{aligned}$$

In the event that $\tilde{x}_\alpha^{[a]}$ and $\tilde{z}_\alpha^{[a]}$ are taken to depend only on t , equation (8) becomes

$$\sum_{\alpha=1}^n \sum_{s=1}^3 \left[-m_\alpha \frac{d\tilde{z}_s^{[a]}}{dt} + m_\alpha \tilde{z}_s^{[a]} \right] \delta_s^{[a]} + \sum_{\alpha=1}^n \sum_{s=1}^3 \left[-m_\alpha \frac{d\tilde{z}_s^{[a]}}{dt} - m_\alpha \left(G \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^n m_\beta \frac{\tilde{z}_s^{[a]} - \tilde{z}_s^{[\beta]}}{\left[\sum_k (\tilde{x}_k^{[a]} - \tilde{x}_k^{[\beta]})^2 \right]^{3/2}} - F_s(\tilde{x}_i^{[a]}, t) \right) \right] \delta_{3+s}^{[a]} = 0 \quad (9)$$

Inasmuch as the coefficients of $\delta_s^{[a]}$ and $\delta_{3+s}^{[a]}$ are independent of z_r and Z_r in (9) the left member of (9) can vanish only if the coefficients of the delta functions are all zero. But this means that we have

$$\begin{cases} \frac{d\tilde{x}_i^{[a]}}{dt} = \tilde{z}_i^{[a]}, \\ \frac{d\tilde{z}_s^{[a]}}{dt} = -G \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^n m_\beta \frac{\tilde{z}_s^{[a]} - \tilde{z}_s^{[\beta]}}{\left[\sum_k (\tilde{x}_k^{[a]} - \tilde{x}_k^{[\beta]})^2 \right]^{3/2}} + F_s(\tilde{x}_i^{[a]}, t), \end{cases}$$

which are the standard equations for the n -body problem.

Numerous problems are more advantageously treated in some other coordinate system than rectangular. We shall consider several examples where cylindrical coordinates are most natural. Let these coordinates be r, θ, z and the components of velocity in these coordinate directions be $R, \Theta, \dot{z}$. Liouville's Equation in cylindrical coordinates has the form (1, p. 187)

$$\begin{aligned} \frac{\partial f}{\partial t} + R \frac{\partial f}{\partial r} + \frac{\Theta}{r} \frac{\partial f}{\partial \theta} + \dot{z} \frac{\partial f}{\partial z} + \left(\frac{\Theta^2}{r} - \frac{\partial U}{\partial r} + F_r \right) \frac{\partial f}{\partial R} + \left(-\frac{R\Theta}{r} - \frac{1}{r} \frac{\partial U}{\partial \theta} + F_\theta \right) \frac{\partial f}{\partial \Theta} \\ + \left(-\frac{\partial U}{\partial z} + F_z \right) \frac{\partial f}{\partial \dot{z}} = 0. \end{aligned} \quad (10)$$

The density distribution function representing the n-body system in this set of coordinates then is

$$f = \sum_{\alpha=1}^n m_{\alpha} \delta(r - \tilde{r}^{[\alpha]}, \theta - \tilde{\theta}^{[\alpha]}, \tilde{z} - \tilde{z}^{[\alpha]}, R - \tilde{R}^{[\alpha]}, \Theta - \tilde{\Theta}^{[\alpha]}, Z - \tilde{Z}^{[\alpha]})$$

and the self-gravitation potential is

$$U = -G \sum_{\alpha=1}^n \frac{m_{\alpha}}{\sqrt{r^2 + \tilde{r}^{[\alpha]2} - 2r\tilde{r}^{[\alpha]}\cos(\theta - \tilde{\theta}^{[\alpha]}) + (\tilde{z} - \tilde{z}^{[\alpha]})^2}}.$$

Example: Two-body problem, circular orbits. In this case $n = 2$ and the external force is zero. Let $\tilde{r}_1^{[1]} = r_1, \tilde{r}_2^{[2]} = r_2, \tilde{\theta}^{[1]} = \tilde{\theta}^{[1]}(t), \tilde{\theta}^{[2]} = \tilde{\theta}^{[2]}(t), \tilde{\Theta}^{[1]} = \omega r_1, \tilde{\Theta}^{[2]} = \omega r_2, \tilde{z}^{[1]} = \tilde{z}^{[2]} = \tilde{R}^{[1]} = \tilde{R}^{[2]} = \tilde{Z}^{[1]} = \tilde{Z}^{[2]} = 0$, and take r_1, r_2, ω to be constants. This means we are assuming a density distribution function of the form

$$f = \sum_{\alpha=1}^2 m_{\alpha} \delta(r - r_{\alpha}, \theta - \tilde{\theta}^{[\alpha]}(t), \tilde{z}, R, \Theta - \omega r, Z).$$

Substituting this into (10) and making use of the relation $g(r)\delta(r-a) = g(a)\delta(r-a)$ gives the equation

$$\begin{aligned} & -m_1 \frac{d\tilde{\theta}^{[1]}}{dt} \delta_2^{[1]} - m_2 \frac{d\tilde{\theta}^{[2]}}{dt} \delta_2^{[2]} + m_1 \omega \delta_2^{[1]} + m_2 \omega \delta_2^{[2]} + m_1 (\omega^2 r_1 - \frac{\partial \tilde{U}}{\partial r}) \delta_4^{[1]} \\ & + m_2 (\omega^2 r_2 - \frac{\partial \tilde{U}}{\partial r}) \delta_4^{[2]} - \frac{m_1}{r_1} \frac{\partial \tilde{U}}{\partial \theta} \delta_5^{[1]} - \frac{m_2}{r_2} \frac{\partial \tilde{U}}{\partial \theta} \delta_5^{[2]} - m_1 \frac{\partial \tilde{U}}{\partial \tilde{z}} \delta_6^{[1]} - m_2 \frac{\partial \tilde{U}}{\partial \tilde{z}} \delta_6^{[2]} = 0. \quad (11) \end{aligned}$$

The subscripts on δ again mean partial derivatives as in the previous work. For the potential function terms we have

$$\left\{ \begin{aligned} \frac{\partial \tilde{U}}{\partial r} \delta_4^{[1]} &= G m_2 \frac{\lambda_1 - \lambda_2 \cos(\tilde{\theta}^{[1]} - \tilde{\theta}^{[2]})}{[\lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2 \cos(\tilde{\theta}^{[1]} - \tilde{\theta}^{[2]})]^{3/2}} \delta_4^{[1]}, \\ \frac{\partial \tilde{U}}{\partial r} \delta_4^{[2]} &= G m_1 \frac{\lambda_2 - \lambda_1 \cos(\tilde{\theta}^{[2]} - \tilde{\theta}^{[1]})}{[\lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2 \cos(\tilde{\theta}^{[2]} - \tilde{\theta}^{[1]})]^{3/2}} \delta_4^{[2]}, \\ \frac{\partial \tilde{U}}{\partial \theta} \delta_5^{[1]} &= G m_2 \frac{\lambda_1 \lambda_2 \sin(\tilde{\theta}^{[1]} - \tilde{\theta}^{[2]})}{[\lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2 \cos(\tilde{\theta}^{[1]} - \tilde{\theta}^{[2]})]^{3/2}} \delta_5^{[1]}, \\ \frac{\partial \tilde{U}}{\partial \theta} \delta_5^{[2]} &= G m_1 \frac{\lambda_2 \lambda_1 \sin(\tilde{\theta}^{[2]} - \tilde{\theta}^{[1]})}{[\lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2 \cos(\tilde{\theta}^{[2]} - \tilde{\theta}^{[1]})]^{3/2}} \delta_5^{[2]}, \\ \frac{\partial \tilde{U}}{\partial x} \delta_6^{[1]} &= \frac{\partial \tilde{U}}{\partial x} \delta_6^{[2]} = 0. \end{aligned} \right.$$

Consequently, (11) takes on the form

$$\begin{aligned} & (-m_1 \frac{d\tilde{\theta}^{[1]}}{dt} + m_1 \omega) \delta_2^{[1]} + (-m_2 \frac{d\tilde{\theta}^{[2]}}{dt} + m_2 \omega) \delta_2^{[2]} + m_1 \left(\omega^2 \lambda_1 - G m_2 \frac{\lambda_1 - \lambda_2 \cos(\tilde{\theta}^{[1]} - \tilde{\theta}^{[2]})}{[\lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2 \cos(\tilde{\theta}^{[1]} - \tilde{\theta}^{[2]})]^{3/2}} \right) \delta_4^{[1]} \\ & + m_2 \left(\omega^2 \lambda_2 - G m_1 \frac{\lambda_2 - \lambda_1 \cos(\tilde{\theta}^{[2]} - \tilde{\theta}^{[1]})}{[\lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2 \cos(\tilde{\theta}^{[2]} - \tilde{\theta}^{[1]})]^{3/2}} \right) \delta_4^{[2]} - m_1 m_2 G \frac{\lambda_2 \sin(\tilde{\theta}^{[1]} - \tilde{\theta}^{[2]})}{[\lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2 \cos(\tilde{\theta}^{[1]} - \tilde{\theta}^{[2]})]^{3/2}} \delta_5^{[1]} \\ & - m_2 m_1 G \frac{\lambda_1 \sin(\tilde{\theta}^{[2]} - \tilde{\theta}^{[1]})}{[\lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2 \cos(\tilde{\theta}^{[2]} - \tilde{\theta}^{[1]})]^{3/2}} \delta_5^{[2]} = 0, \end{aligned} \quad (12)$$

As a review of the notation note that

$$\begin{aligned} \delta_2^{[1]} &\equiv \frac{\partial \delta(r - \lambda_1, \theta - \tilde{\theta}^{[1]}, z, R, \Theta - \omega t, \tilde{z})}{\partial (\theta - \tilde{\theta}^{[1]})}, \\ \delta_4^{[2]} &\equiv \frac{\partial \delta(r - \lambda_2, \theta - \tilde{\theta}^{[2]}, z, R, \Theta - \omega t, \tilde{z})}{\partial R}, \quad \text{etc.} \end{aligned}$$

The delta function manipulations used in deriving (12) have reduced it to a point where the coefficients of all the delta functions are independent of $r, \theta, z, R, \Theta, Z$ and so these terms are essentially independent of each other. Consequently, in order for (12) to be satisfied these coefficients must all vanish. From the $\delta_1^{[1]}$ and $\delta_2^{[1]}$ terms it is apparent that

$$\begin{cases} \tilde{\theta}^{[1]} = \omega t + c_1, \\ \tilde{\theta}^{[2]} = \omega t + c_2. \end{cases}$$

However, from the δ_5 terms we see that the relation $\tilde{\theta}^{[1]} - \tilde{\theta}^{[2]} = 0$ or π is necessary. We choose π and take $\tilde{\theta}^{[1]} = \omega t$, $\tilde{\theta}^{[2]} = \omega t + \pi$. The substitution of these values into the δ_4 coefficients then gives

$$\begin{cases} \omega^2 r_1 - G m_2 \frac{r_1 + r_2}{(r_1 + r_2)^3} = 0, \\ \omega^2 r_2 - G m_1 \frac{r_1 + r_2}{(r_1 + r_2)^3} = 0. \end{cases}$$

From this we find that $m_2 r_2 = m_1 r_1$ and that $\omega^2 = \frac{G m_1}{r_2 (r_1 + r_2)^2} = \frac{G m_2}{r_1 (r_1 + r_2)^2}$. The expressions for ω can be combined into a symmetric form:

$$\omega = \sqrt{\frac{G}{2}} \frac{\sqrt{m_1 r_1 + m_2 r_2}}{\sqrt{r_1 r_2} (r_1 + r_2)}.$$

Thus we have obtained by the Liouville Equation approach the well known result of the existence of a solution to the two-body problem in the form of two particles revolving in circular orbits about their common center of gravity.

Example: Restricted three-body problem. In this instance $n = 3$. Take $\tilde{r}^{[\alpha]}$, $\tilde{\theta}^{[\alpha]}$, etc., for $\alpha = 1, 2$ to be as they were found in the

previous example. We shall try for the straight line ("syzygy") solution. Consequently, we take $\tilde{h}^{[3]} = h_3$ (constant), $\delta^{[3]} = \omega t$, $\Theta^{[3]} = h\omega$, $\tilde{x}^{[3]} = \tilde{R}^{[3]} = \tilde{Z}^{[3]} = 0$. Write $f = f_0 + f_1$ where $f_0 = \sum_{\alpha=1}^3 m_\alpha \delta^{[\alpha]}$ and $f_1 = m_3 \delta^{[3]}$. This partitioning of the density distribution function divides it into a part, f_0 , governing the motion of the two principal bodies (which function we have ascertained in the previous example satisfies Liouville's Equation) and a part, f_1 , connected with the motion of the third particle. Since f_0 has been decided upon, it is f_1 that has to be determined and Liouville's Equation (10) becomes an equation in f_1 somewhat simplified by the fact that f_0 itself satisfies (10). However, (10) despite its appearance is non-linear due to the presence of f in U . To indicate this we may write $U \equiv U(f) = U(f_0 + f_1)$. But the functional relation of U on f is essentially that of integration, which is a linear operator. This means that $U(f_0 + f_1) = U(f_0) + U(f_1)$. By taking note of all these observations we find the equation for f_1 to be

$$\begin{aligned} \frac{\partial f_1}{\partial t} + R \frac{\partial f_1}{\partial h} + \frac{\Theta}{h} \frac{\partial f_1}{\partial \theta} + \tilde{Z} \frac{\partial f_1}{\partial z} + \left(\frac{\Theta^2}{h} - \frac{\partial U(f_0)}{\partial h} \right) \frac{\partial f_1}{\partial R} - \frac{\partial U(f_1)}{\partial h} \frac{\partial f_0}{\partial R} - \frac{\partial U(f_1)}{\partial h} \frac{\partial f_1}{\partial R} \\ + \left(-\frac{R\Theta}{h} - \frac{1}{h} \frac{\partial U(f_0)}{\partial \theta} \right) \frac{\partial f_1}{\partial \Theta} - \frac{1}{h} \frac{\partial U(f_1)}{\partial \theta} \frac{\partial f_0}{\partial \Theta} - \frac{1}{h} \frac{\partial U(f_1)}{\partial \theta} \frac{\partial f_1}{\partial \Theta} - \frac{\partial U(f_1)}{\partial z} \frac{\partial f_0}{\partial z} \\ - \frac{\partial U(f_0)}{\partial z} \frac{\partial f_1}{\partial z} - \frac{\partial U(f_1)}{\partial z} \frac{\partial f_1}{\partial z} = 0. \end{aligned} \quad (13)$$

The use of the delta function representations for f_0 and f_1 cause (13) to reduce to

$$\begin{aligned} \left(m_3 \omega^2 h_3 - G m_1 m_3 \frac{h_3 + h_2}{|h_3 + h_2|^3} - G m_1 m_3 \frac{h_3 - h_1}{|h_3 - h_1|^3} \right) \delta_4^{[3]} \\ - G m_1 m_3 \frac{h_1 - h_3}{|h_1 - h_3|^3} \delta_4^{[1]} - G m_2 m_3 \frac{h_2 + h_3}{|h_2 + h_3|^3} \delta_4^{[2]} = 0. \end{aligned} \quad (14)$$

Equating the coefficient of $\delta_4^{[1]}$ to zero gives the classical equation for r_3 for the straight line solution to the restricted three-body problem:

$$\omega^2 r_3 - G m_2 \frac{r_3 + r_2}{|r_3 + r_2|^3} - G m_1 \frac{r_3 - r_1}{|r_3 - r_1|^3} = 0.$$

Note, though, that the remaining two terms in (14), the $\delta_4^{[1]}$ and $\delta_4^{[2]}$ terms, cannot be made to vanish. This implies that our assumptions as to the make-up of f_0 and f_1 were too restrictive to allow for an exact solution to the three-body problem. What we have found must then be regarded as an approximate solution, but in what sense the approximation is good is not immediately apparent from (14). We can, however, formulate a modified problem for which our f_1 is an exact solution. This would be done by assuming the masses of the first two bodies so great in comparison with that of the third that the effect of the third's mass on the motion of the first two is negligible (the "restricted three-body problem"). In the Liouville Equation approach this is equivalent to regarding $U(f_0)$ as a given external potential and setting the explicit f_0 in (13) (i.e. $\frac{\partial f_0}{\partial R}, \frac{\partial f_0}{\partial \theta}, \frac{\partial f_0}{\partial z}$) to zero. That is to say, the density distribution function f is merely that for one particle moving in an external force field set up by the unaffected two revolving bodies. Under this modification the worrisome last two terms in (14) do not appear at all while the first remains unaltered.

III. Improvement on Approximate Orbit:

This section is devoted to a discussion of a certain procedure for determining corrections to a given approximate orbit of a particle or system of particles in an external field of force. The procedure is

based on the use of a representation of the density distribution function (for the particles) in the form of a series of delta functions and their derivatives. As it turns out, this is essentially equivalent to employing a power series in the differences between the true and approximate values of the coordinates and velocity components.

Suppose the approximate orbit of a single particle is given by $z_r = \hat{z}_r(t)$, $Z_r = \hat{Z}_r(t)$ where the functions $\hat{z}_r(t)$, $\hat{Z}_r(t)$ are taken as known. Suppose further that the actual solution is representable in the form

$$f = \sum_{n_i=0}^{\infty} \sum_{h_2} \sum_{h_3} \sum_{s_1} \sum_{s_2} \sum_{s_3} c_{n_1, n_2, n_3, s_1, s_2, s_3}(t) \delta^{(n_1, n_2, n_3, s_1, s_2, s_3)} \quad (15)$$

where

$$\delta^{(n_1, n_2, n_3, s_1, s_2, s_3)} \equiv \delta^{(n_1)}(z_1 - \hat{z}_1) \delta^{(n_2)}(z_2 - \hat{z}_2) \delta^{(n_3)}(z_3 - \hat{z}_3) \delta^{(s_1)}(\dot{z}_1 - \hat{\dot{z}}_1) \delta^{(s_2)}(\dot{z}_2 - \hat{\dot{z}}_2) \delta^{(s_3)}(\dot{z}_3 - \hat{\dot{z}}_3)$$

and the superscript means the $r_1 \frac{st}{dt}$ derivative, etc. (Note different use of superscript on the delta functions here than in the preceding section.) On occasion we shall use the convenient abbreviation:

$f = \sum_{n_i, s_i} c_{n_i, s_i}(t) \delta^{(n_i, s_i)}$. Let $\tilde{z}_r(t)$, $\tilde{Z}_r(t)$ be coordinates and velocity components for the exact solution so that the exact solution to Liouville's Equation is to be afforded by $f = m \delta(z_n - \tilde{z}_n, \dot{z}_n - \tilde{\dot{z}}_n)$. Thus, we shall be working with the development of this delta function taken for the point $(\tilde{z}_r, \tilde{Z}_r)$ in the series (15) of derivatives of delta functions taken about the point $(\hat{z}_r, \hat{Z}_r)$.

Expansions in series of the type (15) bear certain formal similarities to expansions in terms of orthogonal functions. In particular, the procedure for determining the coefficients in the expansion of a given function is essentially the same in each case: multiply by certain functions and integrate. In the expansions of type (15) the functions to multiply by are power functions and so the coefficients are being determined by "taking moments".

Suppose we wish to expand a given function $g(z_r, \bar{z}_r)$:

$$g(z_n, \bar{z}_n) = \sum_{n_i, s_i} c_{n_i, s_i} \delta^{(n_1)}(z_1 - \bar{z}_1) \dots \delta^{(s_3)}(z_3 - \bar{z}_3). \quad (16)$$

To determine c_{n_i, s_i} in terms of g we first multiply:

$$\begin{aligned} (z_1 - \bar{z}_1)^{h_1} (z_2 - \bar{z}_2)^{h_2} (z_3 - \bar{z}_3)^{h_3} (z_1 - \bar{z}_1)^{k_1} (z_2 - \bar{z}_2)^{k_2} (z_3 - \bar{z}_3)^{k_3} g(z_n, \bar{z}_n) \\ = \sum_{n_i, s_i} c_{n_i, s_i} (z_1 - \bar{z}_1)^{h_1} \dots (z_3 - \bar{z}_3)^{h_3} \delta^{(n_1)}(z_1 - \bar{z}_1) \dots \delta^{(s_3)}(z_3 - \bar{z}_3), \end{aligned}$$

then integrate:

$$\begin{aligned} \left(\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} (z_1 - \bar{z}_1)^{h_1} \dots (z_3 - \bar{z}_3)^{h_3} g(z_n, \bar{z}_n) dz_1 \dots d\bar{z}_3 \right. \\ \left. = \sum_{n_i, s_i} c_{n_i, s_i} \int_{-\infty}^{\infty} (z_1 - \bar{z}_1)^{h_1} \delta^{(n_1)}(z_1 - \bar{z}_1) dz_1 \dots \int_{-\infty}^{\infty} (z_3 - \bar{z}_3)^{h_3} \delta^{(s_3)}(z_3 - \bar{z}_3) d\bar{z}_3. \right) \quad (17) \end{aligned}$$

The repeated integrals on the right can be evaluated through the use of the formula (6, p. 10)

$$\begin{aligned}
 (z-\bar{z})^h \delta^{(n)}(z-\bar{z}) &= \left[(z-\bar{z})^h \right]_{z=\bar{z}} \delta^{(n)} - h \left[(z-\bar{z})^{h-1} \right]_{z=\bar{z}} \delta^{(n-1)} + \binom{n}{2} \left[h(h-1)(z-\bar{z})^{h-2} \right]_{z=\bar{z}} \delta^{(n-2)} \\
 &\quad - \binom{n}{3} \left[h(h-1)(h-2)(z-\bar{z})^{h-3} \right]_{z=\bar{z}} \delta^{(n-3)} + \dots + (-1)^h \binom{n}{h} h! \delta^{(n-h)} \\
 &= (-1)^h \binom{n}{h} h! \delta^{(n-h)}
 \end{aligned}$$

This is to be zero if $h > n$ which is indeed reflected by the fact that $\binom{n}{h}$, the binomial coefficient, should be defined as being zero for $h > n$.

Consequently, upon integration we find

$$\int_{-\infty}^{\infty} (z-\bar{z})^h \delta^{(n)}(z-\bar{z}) d\bar{z} = (-1)^h \binom{n}{h} h! \int_{-\infty}^{\infty} \delta^{(n-h)}(z-\bar{z}) d\bar{z} = (-1)^h \binom{n}{h} h! \delta_h^n$$

(where δ_h^n is the Kronecker delta: $\delta_h^n = 0$ for $n \neq h$, $\delta_h^n = 1$ for $n = h$) since $\int_{-\infty}^{\infty} \delta(z-\bar{z}) d\bar{z} = 1$, $\int_{-\infty}^{\infty} \delta'(z-\bar{z}) d\bar{z} = \int_{-\infty}^{\infty} \delta''(z-\bar{z}) d\bar{z} = \dots = 0$.

Thereupon the right side of (17) becomes

$$\begin{aligned}
 \sum_{n_i, s_i} c_{n_i, s_i} (-1)^{h_1} \binom{n_1}{h_1} h_1! \delta_{h_1}^{n_1} (-1)^{h_2} \binom{n_2}{h_2} h_2! \delta_{h_2}^{n_2} \dots (-1)^{h_3} \binom{s_3}{k_3} k_3! \delta_{k_3}^{s_3} \\
 = (-1)^{h_1 + \dots + h_3} \binom{h_1}{h_1} \dots \binom{k_3}{k_3} h_1! \dots k_3! c_{h_1, h_2, h_3, k_1, k_2, k_3}
 \end{aligned}$$

This supplies us with the necessary formula for the coefficients in the expansion (16), the formula being

$$C_{h_i, k_i} = \frac{(-1)^{h_1 + \dots + k_3}}{h_1! \dots k_3!} \left(\prod_{i=1}^3 \int_{-\infty}^{\infty} (z_i - \bar{z}_i)^{h_i} \dots (z_3 - \bar{z}_3)^{k_3} g(z_n, \bar{z}_n) dz_1 \dots d\bar{z}_3, \quad (18)$$

and this represents the inversion of (16).

Let us apply this to the case at hand, i.e., $g(z_n, \bar{z}_n) = m \delta(z_n - \tilde{z}_n, \bar{z}_n - \tilde{\bar{z}}_n)$ and $\bar{z}_n = \hat{\bar{z}}_n$, $\tilde{\bar{z}}_n = \hat{\tilde{z}}_n$, and introduce the differences of the true and the approximate:

$$\epsilon_n = \tilde{z}_n - \hat{\bar{z}}_n, \quad \eta_n = \tilde{\bar{z}}_n - \hat{\tilde{z}}_n. \quad (19)$$

In this case the coefficients as given by (18) specialize considerably.

$$\begin{aligned} C_{h_i, k_i} &= \frac{(-1)^{h_1 + \dots + k_3}}{h_1! \dots k_3!} \left(\prod_{i=1}^3 \int_{-\infty}^{\infty} (z_i - \tilde{z}_i + \epsilon_i)^{h_i} \dots (z_3 - \tilde{z}_3 + \eta_3)^{k_3} m \delta(z_n - \tilde{z}_n, \bar{z}_n - \tilde{\bar{z}}_n) dz_1 \dots d\bar{z}_3 \right. \\ &= \frac{m(-1)^{h_1 + \dots + k_3}}{h_1! \dots k_3!} \int_{-\infty}^{\infty} (z_1 - \tilde{z}_1 + \epsilon_1)^{h_1} \delta(z_1 - \tilde{z}_1) dz_1 \dots \int_{-\infty}^{\infty} (z_3 - \tilde{z}_3 + \eta_3)^{k_3} \delta(z_3 - \tilde{z}_3) d\tilde{z}_3. \end{aligned}$$

But $(z - \tilde{z} + \epsilon)^h \delta(z - \tilde{z}) = \epsilon^h \delta(z - \tilde{z})$, implying that the formula for the coefficients in the expansion of $m \delta(z_n - \tilde{z}_n, \bar{z}_n - \tilde{\bar{z}}_n)$ in terms of the

differences ϵ_r , η_r is

$$c_{h_1, h_2, h_3}(t) = \frac{m(t)^{h_1 + \dots + h_3}}{h_1! \dots h_3!} \epsilon_1^{h_1} \epsilon_2^{h_2} \epsilon_3^{h_3} \eta_1^{k_1} \eta_2^{k_2} \eta_3^{k_3}. \quad (20)$$

These coefficients are thus of especial interest to us since they are so simply related to the errors. Those that we actually want are those for which $h_1 + \dots + h_3 = 1$ since (20) tells us that $c_{100000} = -m\epsilon_1$, $c_{010000} = -m\epsilon_2$, etc.

To impose the condition represented by the equation of continuity we substitute (15) into (6) and get first of all

$$\begin{aligned} \sum_{n_i, s_i} \dot{c}_{n_i, s_i} \delta^{(n_i, s_i)} - \sum_{n_i, s_i} c_{n_i, s_i} \sum_p \dot{\tilde{z}}_p \delta^{(n_1, \dots, n_p+1, \dots, s_3)} - \sum_{n_i, s_i} c_{n_i, s_i} \sum_p \dot{\tilde{z}}_p \delta^{(n_1, \dots, s_p+1, \dots, s_3)} \\ + \sum_{n_i, s_i} c_{n_i, s_i} \sum_k \tilde{z}_k \delta^{(n_1, \dots, n_k+1, \dots, s_3)} + \sum_{n_i, s_i} c_{n_i, s_i} \sum_k F_k \delta^{(n_1, \dots, s_k+1, \dots, s_3)} = 0. \end{aligned} \quad (21)$$

The guiding principle behind the handling of this is to reduce the left side to a sum of essentially independent terms. This is accomplished by getting the coefficients of the delta functions to be independent of the phase space variables $z_r, \tilde{z}_r$. Manipulations that bring this about are of the same type used earlier. In particular,

$$\left\{ \begin{aligned} \tilde{z}_k \delta^{(n_1, \dots, n_k+1, \dots, s_3)} \delta^{(s_1)} (\tilde{z}_1 - \tilde{z}_1) \delta^{(s_2)} (\tilde{z}_2 - \tilde{z}_2) \delta^{(s_3)} (\tilde{z}_3 - \tilde{z}_3) &= \tilde{z}_k \delta^{(n_1, \dots, n_k+1, \dots, s_3)} - s_k \delta^{(n_1, \dots, n_k+1, \dots, s_k-1, \dots, s_3)}, \\ F_k \delta^{(n_1, \dots, s_k+1, \dots, s_3)} &= \sum_{\lambda_1=0}^{\lambda_1} \sum_{\lambda_2=0}^{\lambda_2} \sum_{\lambda_3=0}^{\lambda_3} (-1)^{\lambda_1+\lambda_2+\lambda_3} \binom{n_1}{\lambda_1} \binom{n_2}{\lambda_2} \binom{n_3}{\lambda_3} \hat{F}_k \delta^{(n_1-\lambda_1)} \delta^{(n_2-\lambda_2)} \delta^{(n_3-\lambda_3)} \delta^{(s_1, \dots, s_k+1, \dots)}, \\ \hat{F}_k &\equiv \frac{\partial^{\lambda_1+\lambda_2+\lambda_3} F_k}{\partial \tilde{z}_1^{\lambda_1} \partial \tilde{z}_2^{\lambda_2} \partial \tilde{z}_3^{\lambda_3}} \text{ evaluated at } \tilde{z}_i = \hat{\tilde{z}}_i. \end{aligned} \right.$$

Employing these relations and, where necessary, shifting the indices on the various sums so that all sums have the delta function derivatives written as $\delta^{(n_i, s_i)}$ enables us to add all these sums together as one sum on $\delta^{(n_i, s_i)}$ with coefficients independent of z, Z_r . These coefficients must then vanish if the equation is to be satisfied, and so we obtain as the governing equation for $c_{n_i, s_i}(t)$

$$\begin{aligned} \dot{c}_{n_i, s_i} - \sum_{p=1}^3 (\dot{\hat{z}}_p - \hat{Z}_p) c_{\dots, n_p-1, \dots} - \sum_{p=1}^3 \hat{Z}_p c_{\dots, s_p-1, \dots} - \sum_{p=1}^3 (s_p+1) c_{\dots, n_p+1, \dots, s_p+1, \dots} \\ + \sum_{p=1}^3 \sum_{\lambda_1=0}^{\infty} \sum_{\lambda_2=0}^{\infty} \sum_{\lambda_3=0}^{\infty} (-1)^{\lambda_1+\lambda_2+\lambda_3} \binom{n_1+\lambda_1}{\lambda_1} \binom{n_2+\lambda_2}{\lambda_2} \binom{n_3+\lambda_3}{\lambda_3} \hat{F}_{\underbrace{\lambda_1 \lambda_2 \lambda_3}_{\lambda_1 \lambda_2 \lambda_3}} c_{n_1+\lambda_1, n_2+\lambda_2, n_3+\lambda_3, s_1, s_2, s_3-1, \dots} = 0. \end{aligned} \quad (22)$$

It is to be understood in (22) that any c_{n_i, s_i} with negative index is regarded as zero. In a particular problem in practice we may well have $\dot{\hat{z}}_p = \hat{Z}_p$, which simplifies the equation a little. The dot over the letters, of course, stands for the total derivative with respect to the time variable.

An inspection of (22) tells us that ultimately an integration will have to be effected in order to determine c_{n_i, s_i} . In anticipation of this we turn our attention to the constants of integration. These depend on the initial values of the differences of $\hat{z}_n(t)$ and $\hat{Z}_n(t)$, $\hat{Z}_n(t)$ and $\hat{Z}_n(t)$, i.e., on $\epsilon_n(0)$ and $\eta_n(0)$. Consequently, with these numbers specified we get immediately from (20) the initial values of $c_{n_i, s_i}(t)$:

$$c_{n_i, s_i}(0) = \frac{m (-1)^{\lambda_1 + \dots + s_3}}{n_1! \dots s_3!} \epsilon_1^{n_1}(0) \epsilon_2^{s_2}(0) \epsilon_3^{n_3}(0) \eta_1^{s_1}(0) \eta_2^{s_2}(0) \eta_3^{s_3}(0). \quad (23)$$

In the special case that is apt to occur, namely, that the approximate solution, $\hat{z}_r, \hat{z}_r$ have the same initial values as the exact solution $\tilde{z}_r, \tilde{z}_r$ there is considerable simplification: $C_{00000}(0) = m$ and all other $C_{h_i, s_i} = 0$.

Equation (22) is a differential-difference-sum equation in C_{h_i, s_i} and as such can be looked at from many different viewpoints. Perhaps the most natural way is to regard it primarily as a difference equation which may be treated as a recursion relation, with each step in the sequence providing a differential equation. A logical successive approximation scheme based on this view would begin by considering first only those equations in (22) starting with those $\dot{C}_{h_i, s_i}$ whose subscripts add up to one: $h_1 + h_2 + h_3 + s_1 + s_2 + s_3 = 1$ and suppress all terms having indices on the c 's adding up to more than one. As is apparent from (20) this means we are neglecting terms of the second and higher degree in the correction function; that is to say, so far it is a first order theory in the size of the error functions. To obtain a second order theory stick the values of C_{h_i, s_i} with $h_1 + \dots + s_3 = 1$ just found (by integrating the system of equations we have just indicated) into the equations extracted from (22) headed $\dot{C}_{h_i, s_i}$ with $h_1 + \dots + s_3 = 2$, with all terms where this sum is greater than 2 tossed out. Solve this system of differential equations for C_{h_i, s_i} (with $h_1 + \dots + s_3 = 2$) and use these values back in the equations headed $\dot{C}_{h_i, s_i}$ with $h_1 + \dots + s_3 = 1$. Solving this system then gives C_{h_i, s_i} with $h_1 + \dots + s_3 = 1$ to the second order in the correction functions. This process can evidently be continued in theory by picking up the bunch of equations where the sum is 3, etc. A control on the speed of convergence is afforded by such relations as $C_{20000} = \frac{1}{2} \dot{C}_{10000}^2$, etc., which result from (20). In fact, if one chooses he can use these relations to determine the first approximations to C_{h_i, s_i} with $h_1 + \dots + s_3 = 2$ rather than the differential equations for them.

There are, of course, other procedures that can be devised for obtaining approximate solutions to (22). We give one that draws heavily on the special form of the equation. We treat the one-dimensional problem first and then extend the results to the three-dimensions.

$$\text{Let } G_{n,\lambda} = \dot{\hat{Z}} - \hat{F}, \quad \lambda = 0, \\ = (-1)^{\lambda+1} \binom{n+\lambda}{\lambda} \hat{F}_{x \dots x}, \quad \lambda = 1, 2, 3, \dots$$

The one-dimensional equation with this simplification in notation takes on the appearance

$$\dot{c}_{n,s} - (\dot{\hat{z}} - \hat{Z})c_{n-1,s} - (s+1)c_{n-1,s+1} = \sum_{\lambda=0}^{\infty} G_{n,\lambda} c_{n+\lambda,s-1}. \quad (24)$$

This can be looked upon as a sum equation in the $c_{n+\lambda,s-1}$ on the right with the left side regarded as known, say by assuming some approximate values for them. In that event, it is of the "semi-reduced" or "semi-normal" type (7, p. 535). However, we examine (24) from another point of view. One way is to integrate formally and then use successive approximations. This in effect inverts the $\frac{d}{dt}$ operator. Convergence properties are apt to be improved, however, if a larger portion of the differential-difference operator that transforms $c_{n,s}$ on the left is inverted. We proceed to do this for the operator T where $Tc_{n,s} = \dot{c}_{n,s} - (s+1)c_{n-1,s+1}$. Consequently, we write

$$\dot{c}_{n,s} - (s+1)c_{n-1,s+1} = \Phi_{n,s}(t) \quad (25)$$

where

$$\Phi_{r,s}(t) \equiv (\dot{\hat{z}} - \hat{z}) c_{r-1,s} + \sum_{\lambda=0}^{\infty} G_{r,\lambda} c_{r+\lambda,s-1}.$$

The simplifying feature of (25) is the fact that the sum of the indices on each of the two terms on the left is the same, namely $r + s$. From the difference equation point of view this means that Laplace's substitution ((7, p. 427)) is effective:

$$\begin{cases} r = n + s, \\ \sigma = s, \end{cases} \quad \begin{cases} n = r - \sigma, \\ s = \sigma. \end{cases}$$

This throws (25) over into

$$\dot{c}_{r-\sigma,\sigma} - (\sigma+1)c_{r-\sigma-1,\sigma+1} = \Phi_{r-\sigma,\sigma}(t) \quad (26)$$

wherein it is apparent that as a difference equation this is essentially an equation in σ . Emphasize this by writing $c_{r-\sigma,\sigma} \equiv u_{\sigma}$ and

$$\Phi_{r-\sigma,\sigma}(t) \equiv \phi_{\sigma}(t).$$

Thus the equation to be examined is

$$\dot{u}_{\sigma} - (\sigma+1)u_{\sigma+1} = \phi_{\sigma}(t). \quad (27)$$

The associated homogeneous equation to (27) (i.e. the equation with $\phi(t)$ replaced by zero) has the solution ((7, p. 327))

$$\beta \frac{\gamma^\sigma e^{\gamma t}}{\Gamma(\sigma+1)}$$

where β is a periodic function of σ of period one; but since we are concerned with only integral values of σ we have β a constant. The variation of parameters method then suffices for the inhomogeneous equation (27). Assume

$$u_\sigma(t) = \frac{\gamma^\sigma e^{\gamma t}}{\Gamma(\sigma+1)} \omega_\sigma(t) \quad (28)$$

and substitute it into (27), thereby deriving the equation for $\omega_\sigma(t)$ as

$$\dot{\omega}_\sigma - \gamma \omega_{\sigma+1} + \gamma \omega_\sigma = \frac{\Gamma(\sigma+1)}{\gamma^\sigma e^{\gamma t}} \phi_\sigma(t). \quad (29)$$

Introduce the "shifting" operator E , i.e. $E \omega_\sigma \equiv \omega_{\sigma+1}$, and regard it as a constant, so that the differential-difference equation (29) takes on the appearance of a differential equation alone:

$$\dot{\omega}_\sigma - \gamma(E-1)\omega_\sigma = \frac{\Gamma(\sigma+1)}{\gamma^\sigma e^{\gamma t}} \phi_\sigma(t). \quad (30)$$

This linear differential equation with constant coefficients has as solution

$$w_{\sigma} = e^{-\gamma t} \int_0^t e^{\gamma E(t-\tau)} \frac{\Gamma(\sigma+1)}{\gamma^{\sigma}} \phi_{\sigma}(\tau) d\tau + e^{\gamma(E-1)t} C_{\sigma}, \quad (31)$$

where C_{σ} is independent of t . Observe that, although $\Gamma(\sigma+1)\gamma^{-\sigma}$ is independent of τ , it is nonetheless left in the integrand since it must lie to the right of $e^{\gamma E(t-\tau)}$ which operates on it due to the presence of E . The solution to (27) then by virtue of (28) and (31) has the form

$$u_{\sigma} = \frac{\gamma^{\sigma}}{\Gamma(\sigma+1)} \int_0^t e^{\gamma E(t-\tau)} \frac{\Gamma(\sigma+1)}{\gamma^{\sigma}} \phi_{\sigma}(\tau) d\tau + \frac{\gamma^{\sigma}}{\Gamma(\sigma+1)} e^{\gamma Et} C_{\sigma}. \quad (32)$$

The effect of the operator $e^{\gamma E(t-\tau)}$ has to be determined to make this usable. This is most easily accomplished by employing the Maclaurin expansion for the exponential function and making use of the property $E^n f(\sigma) = f(\sigma+n)$. We have

$$\begin{aligned} u_{\sigma} &= \frac{\gamma^{\sigma}}{\Gamma(\sigma+1)} \int_0^t \sum_{k=0}^{\infty} \frac{1}{k!} \gamma^k (t-\tau)^k E^k \frac{\Gamma(\sigma+1)}{\gamma^{\sigma}} \phi_{\sigma}(\tau) d\tau \\ &\quad + \frac{\gamma^{\sigma}}{\Gamma(\sigma+1)} \sum_{k=0}^{\infty} \frac{1}{k!} \gamma^k t^k E^k C_{\sigma}, \\ &= \frac{\gamma^{\sigma}}{\Gamma(\sigma+1)} \int_0^t \sum_{k=0}^{\infty} \frac{1}{k!} \gamma^k (t-\tau)^k \frac{\Gamma(\sigma+1+k)}{\gamma^{\sigma+k}} \phi_{\sigma+k}(\tau) d\tau \\ &\quad + \frac{\gamma^{\sigma}}{\Gamma(\sigma+1)} \sum_{k=0}^{\infty} \frac{1}{k!} \gamma^k t^k C_{\sigma+k}, \\ &= \int_0^t \sum_{k=0}^{\infty} \binom{\sigma+k}{k} (t-\tau)^k \phi_{\sigma+k}(\tau) d\tau + \frac{\gamma^{\sigma}}{\Gamma(\sigma+1)} \sum_{k=0}^{\infty} \frac{1}{k!} (\gamma t)^k C_{\sigma+k}. \end{aligned}$$

The symbol $\binom{\sigma+k}{k}$, as before, stands for the binomial coefficients:
 $\binom{\sigma+k}{k} = \frac{(\sigma+k)!}{\sigma! k!}$. The quantities C_σ , which are independent of t but dependent on σ , stemmed from working on the equation (27). If we return to (26), the equation involving both ρ and σ but with ρ playing no effective role, we see that we can take $C_{\sigma+k}$ as $C_{\rho, \sigma+k}$. Consequently, upon retrieving the variables r and s we find as general solution to (25)

$$c_{n,s}(t) = \sum_{k=0}^{\infty} \binom{s+k}{k} [t^k * \Phi_{n-k, s+k}(t)] + \frac{\gamma^s}{\Gamma(s+1)} \sum_{k=0}^{\infty} \frac{1}{k!} (\gamma t)^k C_{n+s, s+k}, \quad (33)$$

assuming that term by term integration is justified. For the τ integral in this formula we have used the convolution notation ((6, p. 31))

$$t^k * \Phi_{n-k, s+k}(t) \equiv \int_0^t (t-\tau)^k \Phi_{n-k, s+k}(\tau) d\tau.$$

The initial conditions will serve to determine the constants $C_{n,s}$. Set $t = 0$ in (33) and get

$$c_{n,s}(0) = \frac{\gamma^s}{\Gamma(s+1)} C_{n+s, s}$$

which implies that

$$C_{n+s, s+k} = \frac{\Gamma(s+k+1)}{\gamma^{s+k}} c_{n-k, s+k}(0). \quad (34)$$

We incorporate these values into (33) and obtain the final result:

$$c_{n,s}(t) = \sum_{k=0}^n \binom{s+k}{k} \left[t^k * \left\{ (\dot{\hat{Z}} - \hat{F}) c_{n-k, s+k}(t) + \sum_{\lambda=0}^{\infty} G_{n-k, \lambda} c_{n-k+\lambda, s+k-1}(t) \right\} \right] \\ + \sum_{k=0}^n \binom{s+k}{k} c_{n-k, s+k}(0) \cdot t^k. \quad (35)$$

In the event that the exact and approximate orbits have the same initial values the final series in (35), i.e. the one involving $c_{n-k, s+k}(0)$, drops out completely, if we take the range on r and s to be $n+s \geq 1$. This is no real restriction since $c_{00}(t)$ is the mass and therefore considered known. The upper limit on the summation signs in (35) is r instead of ∞ because $c_{n-k, s+k}$ and $G_{n-k, \lambda}$ vanish for $k > r$. Observe that this final result (35) does not involve the arbitrary constant γ introduced as an auxiliary to enable us to solve (25).

While (35) can be said to solve (25) if the right member of (25) is taken as given, actually the right side involves the unknowns, the $c_{n,s}$. Thus, the inversion of the operator has in effect thrown the differential-difference equation (24) over into an integral equation but as such it is well adapted for approximation methods, say the method of iteration.

If one wishes to stick to a first order theory the equations are obtained from (35) by setting $c_{n,s}(t) = 0$ for $n+s \geq 2$. These equations are

$$\begin{cases} c_{01}(t) = \int_0^t [(\dot{\hat{Z}} - \hat{F}) c_{00} + \hat{F}_z c_{10}] d\tau + c_{01}(0), \\ c_{10}(t) = \int_0^t (\dot{\hat{Z}} - \hat{Z}) c_{00} d\tau + \int_0^t (t-\tau) [(\dot{\hat{Z}} - \hat{F}) c_{00} + \hat{F}_z c_{10}] d\tau + c_{10}(0) + t c_{01}(0). \end{cases}$$

As mentioned above, the terms $c_{01}(0)$, $c_{10}(0)$, $t c_{01}(0)$ all disappear if the exact and approximate orbits have the same initial values.

For a second order theory, assume $c_{\lambda s}(t) = 0$ if $\lambda + s \geq 3$. The equations in this case are

$$\left\{ \begin{aligned} c_{01}(t) &= \int_0^t [(\dot{\hat{Z}} - \hat{F})c_{00} + \hat{F}_{\bar{z}}c_{10} - \hat{F}_{\bar{z}\bar{z}}c_{20}]d\tau + c_{01}(0), \\ c_{10}(t) &= \int_0^t (\dot{\hat{z}} - \hat{Z})c_{00}d\tau + \int_0^t (t-\tau)[(\dot{\hat{Z}} - \hat{F})c_{00} + \hat{F}_{\bar{z}}c_{10} - \hat{F}_{\bar{z}\bar{z}}c_{20}]d\tau + c_{10}(0) + t c_{01}(0), \\ c_{02}(t) &= \int_0^t [(\dot{\hat{Z}} - \hat{F})c_{01} + \hat{F}_{\bar{z}}c_{11}]d\tau + c_{02}(0), \\ c_{11}(t) &= \int_0^t [(\dot{\hat{z}} - \hat{Z})c_{01} + (\dot{\hat{Z}} - \hat{F})c_{10} + 2\hat{F}_{\bar{z}}c_{20}]d\tau \\ &\quad + 2 \int_0^t (t-\tau)[(\dot{\hat{Z}} - \hat{F})c_{01} + \hat{F}_{\bar{z}}c_{11}]d\tau + c_{11}(0) + 2t c_{02}(0), \\ c_{20}(t) &= \int_0^t (\dot{\hat{z}} - \hat{Z})c_{10}d\tau + \int_0^t (t-\tau)[(\dot{\hat{z}} - \hat{Z})c_{01} + (\dot{\hat{Z}} - \hat{F})c_{10} + 2\hat{F}_{\bar{z}}c_{20}]d\tau \\ &\quad + \int_0^t (t-\tau)^2[(\dot{\hat{Z}} - \hat{F})c_{01} + \hat{F}_{\bar{z}}c_{11}]d\tau + c_{20}(0) + t c_{11}(0) + t^2 c_{02}(0). \end{aligned} \right.$$

The extension from one dimension to three dimensions is almost immediate. After the substitutions $\eta_i + s_i = \rho_i$, $s_i = \sigma_i$, a separation of variables does the work as before. In place of (25) we now consider (22) written as

$$\dot{c}_{\eta_1, \eta_2, \eta_3, s_1, s_2, s_3}(t) - \sum_{p=1}^3 (s_p + 1) c_{\eta_1, \dots, \eta_p - 1, \dots, \eta_3, s_1, \dots, s_p + 1, \dots, s_3} = \Phi_{\eta_1, \eta_2, \eta_3, s_1, s_2, s_3}(t).$$

The comparable result to (35) then is

$$\begin{aligned} c_{\eta_1, \eta_2, \eta_3, s_1, s_2, s_3}(t) &= \sum_{k_1=0}^{\eta_1} \sum_{k_2=0}^{\eta_2} \sum_{k_3=0}^{\eta_3} \binom{s_1+k_1}{k_1} \binom{s_2+k_2}{k_2} \binom{s_3+k_3}{k_3} \left[t^{k_1+k_2+k_3} * \Phi_{\eta_1-k_1, \eta_2-k_2, \eta_3-k_3, s_1+k_1, s_2+k_2, s_3+k_3} \right] \\ &\quad + \sum_{k_1=0}^{\eta_1} \sum_{k_2=0}^{\eta_2} \sum_{k_3=0}^{\eta_3} \binom{s_1+k_1}{k_1} \binom{s_2+k_2}{k_2} \binom{s_3+k_3}{k_3} t^{k_1+k_2+k_3} c_{\eta_1-k_1, \eta_2-k_2, \eta_3-k_3, s_1+k_1, s_2+k_2, s_3+k_3}(0). \end{aligned}$$

The preceding work on the improvement of an approximate orbit for a single particle suffices as well for a system of n particles,--one has but to let the indices 1, 2, 3 refer to the first particle, 4, 5, 6 to the second and so on, letting the range on p and on i in r_i and s_i be 1 to $3n$ instead of 1 to 3 in the foregoing formulae. If, however, one prefers to keep these ranges 1 to 3 and indicate the different particles by a separate index, say α , with range 1 to n it is just as easy to obtain the necessary formulae and equations. In this latter approach we indicate the approximate orbits of the n particles by $\mathbf{r}_n^i = \hat{\mathbf{r}}_n^{[\alpha]}(t)$, $\mathbf{Z}_n = \hat{\mathbf{Z}}_n^{[\alpha]}(t)$ where $r = 1, 2, 3$ and $\alpha = 1, 2, \dots, n$. The exact solution we assume to be representable in the form

$$f = \sum_{\alpha=1}^n \sum_{\eta_1=0}^{\infty} \sum_{\eta_2} \sum_{\eta_3} \sum_{s_1} \sum_{s_2} \sum_{s_3} c_{\eta_1 \eta_2 \eta_3 s_1 s_2 s_3}^{[\alpha]}(t) \delta^{(\eta_1)}(\mathbf{z}_1 - \hat{\mathbf{z}}_1^{[\alpha]}) \dots \delta^{(s_3)}(\mathbf{Z}_3 - \hat{\mathbf{Z}}_3^{[\alpha]}), \quad (36)$$

for which we employ the abbreviated notation

$$f = \sum_{\alpha} \sum_{\eta_i s_i} c_{\eta_i s_i}^{[\alpha]}(t) \delta^{[\alpha]}(\eta_1) \dots \delta^{[\alpha]}(s_3)$$

If the actual trajectories are given as $\mathbf{r}_n = \tilde{\mathbf{r}}_n^{[\alpha]}(t)$, $\mathbf{Z}_n = \tilde{\mathbf{Z}}_n^{[\alpha]}(t)$, then the exact solution also has the representation

$$f = \sum_{\alpha=1}^n m_{\alpha} \delta(\mathbf{z}_1 - \tilde{\mathbf{z}}_1^{[\alpha]}) \delta(\mathbf{z}_2 - \tilde{\mathbf{z}}_2^{[\alpha]}) \delta(\mathbf{z}_3 - \tilde{\mathbf{z}}_3^{[\alpha]}) \delta(\mathbf{Z}_1 - \tilde{\mathbf{Z}}_1^{[\alpha]}) \delta(\mathbf{Z}_2 - \tilde{\mathbf{Z}}_2^{[\alpha]}) \delta(\mathbf{Z}_3 - \tilde{\mathbf{Z}}_3^{[\alpha]}).$$

The differences between the exact and approximate values for the coordinates and velocity components are written as

$$\epsilon_n^{[\alpha]} = \tilde{\mathbf{r}}_n^{[\alpha]} - \hat{\mathbf{r}}_n^{[\alpha]}, \quad \eta_n = \tilde{\mathbf{Z}}_n^{[\alpha]} - \hat{\mathbf{Z}}_n^{[\alpha]},$$

whereupon we obtain as counterpart to (20) the formula for the coefficients in (36) for the expansion of the exact solution distribution function

$$c_{h_i, k_i}^{[\alpha]}(t) = \frac{m_\alpha (-1)^{h_i + \dots + k_3}}{h_i! \dots k_3!} \epsilon_1^{[\alpha] h_i} \dots \eta_3^{[\alpha] k_3} \quad (37)$$

Working as before leads to the fundamental equation, the counterpart of (22),

$$\begin{aligned} \dot{c}_{n_i, s_i}^{[\alpha]} - \sum_{p=1}^3 \left(\dot{\hat{z}}_p^{[\alpha]} - \hat{z}_p^{[\alpha]} \right) c_{\dots, n_p-1, \dots}^{[\alpha]} - \sum_{p=1}^3 \hat{z}_p^{[\alpha]} c_{\dots, s_p-1, \dots}^{[\alpha]} - \sum_{p=1}^3 (s_p+1) c_{\dots, n_p-1, \dots, s_p+1, \dots}^{[\alpha]} \\ + \sum_{p=1}^3 \sum_{\lambda_i=0}^{\infty} \sum_{\lambda_2} \sum_{\lambda_3} (-1)^{\lambda_1+\lambda_2+\lambda_3} \binom{n_1+\lambda_1}{\lambda_1} \binom{n_2+\lambda_2}{\lambda_2} \binom{n_3+\lambda_3}{\lambda_3} \hat{F}_p^{[\alpha]} \underbrace{c_{n_1+\lambda_1, n_2+\lambda_2, n_3+\lambda_3, \dots, s_p-1, \dots}^{[\alpha]}}_{\substack{\lambda_1 \quad \lambda_2 \quad \lambda_3}} = 0 \end{aligned} \quad (38)$$

Here as in (22) the notation $c_{\dots, n_p-1, \dots}^{[\alpha]}$ means that the subscripts are the same as for $c_{n_1, n_2, n_3, s_1, s_2, s_3}^{[\alpha]}$ with the exception of the subscript n_p which has been changed to n_p-1 . Also, as before, any $c_{n_i, s_i}^{[\alpha]}$ with a negative index is to be thought of as zero.

If the self-gravitation of the particles forms an appreciable part of the force field then it must be included in the $\hat{F}_p^{[\alpha]}$ term in (38). It has a reasonably simple formulation involving the $c_{n_i, s_i}^{[\alpha]}$ which we proceed to derive.

Integrate over the velocity space to obtain the position space density $N(\bar{\mathbf{z}}, t)$:

$$N = \iiint_{-\infty}^{\infty} \sum_{\alpha} \sum_{h_i, s_i} c_{h_i, s_i}^{[\alpha]}(t) \delta^{(h_1)} \dots \delta^{(s_3)} d\mathbf{z}_1 d\mathbf{z}_2 d\mathbf{z}_3,$$

$$= \sum_{\alpha} \sum_{h_i} c_{h_1, h_2, h_3, 000}^{[\alpha]}(t) \delta^{(h_1)}(\bar{\mathbf{z}}_1 - \hat{\bar{\mathbf{z}}}_1^{[\alpha]}) \delta^{(h_2)}(\bar{\mathbf{z}}_2 - \hat{\bar{\mathbf{z}}}_2^{[\alpha]}) \delta^{(h_3)}(\bar{\mathbf{z}}_3 - \hat{\bar{\mathbf{z}}}_3^{[\alpha]}).$$

The $c_{h_1, h_2, h_3, 000}^{[\alpha]}(t)$ involve the masses of the particles, as is evidenced by (37). The gravitation potential per unit mass due to the particles themselves is then

$$U = -G \iiint_{-\infty}^{\infty} \sum_{\alpha} \sum_{h_i} c_{h_1, h_2, h_3, 000}^{[\alpha]}(t) \frac{\delta^{(h_1)}(\bar{\mathbf{z}}_1 - \hat{\bar{\mathbf{z}}}_1^{[\alpha]}) \delta^{(h_2)}(\bar{\mathbf{z}}_2 - \hat{\bar{\mathbf{z}}}_2^{[\alpha]}) \delta^{(h_3)}(\bar{\mathbf{z}}_3 - \hat{\bar{\mathbf{z}}}_3^{[\alpha]})}{\sqrt{(\bar{\mathbf{z}}_1 - \bar{\mathbf{z}}_1)^2 + (\bar{\mathbf{z}}_2 - \bar{\mathbf{z}}_2)^2 + (\bar{\mathbf{z}}_3 - \bar{\mathbf{z}}_3)^2}} d\bar{\mathbf{z}}_1 d\bar{\mathbf{z}}_2 d\bar{\mathbf{z}}_3,$$

$$= -G \sum_{\alpha} \sum_{h_i} c_{h_1, h_2, h_3, 000}^{[\alpha]}(t) (-1)^{h_1+h_2+h_3} \left[\frac{\partial^{h_1+h_2+h_3}}{\partial \bar{\mathbf{z}}_1^{h_1} \partial \bar{\mathbf{z}}_2^{h_2} \partial \bar{\mathbf{z}}_3^{h_3}} \frac{1}{\sqrt{\sum_k (\bar{\mathbf{z}}_k - \bar{\bar{\mathbf{z}}}_k)^2}} \right]_{\bar{\mathbf{z}}_k = \hat{\bar{\mathbf{z}}}_k^{[\alpha]}}.$$

Recall that the convention we have adopted is that it is the negative of the gradient of the potential that is the force vector. When incorporating $-\frac{\partial U}{\partial \bar{\mathbf{z}}_n}$ into F_p we should recall from our earlier treatment of the n-body problem that in the evaluation of F_p at $\bar{\mathbf{z}}_n = \hat{\bar{\mathbf{z}}}_n^{[\alpha]}$ that is assumed in (38) the terms that become not well-defined are to be suppressed.

IV. West Ford Needles-Dispensing:

An example of a problem that can be treated reasonably as degenerate in velocity space but not in position space is that afforded by the West Ford Needles ((10, 11)). The proposal was to put a large number of small needle-shaped particles into circular orbits around the earth to form several thin belts. This presents several dynamics problems of interest, in particular, that of the dispensing of the needles from the original container to spread out into the belt and that of the dispersion effect on the belt by the perturbational forces that may reasonably be assumed to be present. It is not the purpose here to attack these problems exhaustively but rather to provide a sufficiently broad treatment to exhibit the aptness of the weak functions in situations of this general character.

We treat first the problem of the dispensing of the needles from the container. The natural coordinates are the spherical coordinates ρ, θ, φ where θ is the longitude and φ the co-latitude. The components of the linear velocity vector in these respective directions are indicated by P, Θ, Φ . Liouville's Equation for such a coordinate system is

$$\begin{aligned} \frac{\partial f}{\partial t} + P \frac{\partial f}{\partial \rho} + \frac{\Theta}{\rho \sin \varphi} \frac{\partial f}{\partial \theta} + \frac{\Phi}{\rho} \frac{\partial f}{\partial \varphi} + \left(-\frac{\partial U}{\partial \rho} + \frac{\Theta^2}{\rho} + \frac{\Phi^2}{\rho} \right) \frac{\partial f}{\partial P} \\ + \left(-\frac{1}{\rho \sin \varphi} \frac{\partial U}{\partial \theta} - \frac{P\Theta}{\rho} - \frac{\cot \varphi \Phi \Theta}{\rho} \right) \frac{\partial f}{\partial \Theta} + \left(-\frac{1}{\rho} \frac{\partial U}{\partial \varphi} - \frac{P\Phi}{\rho} + \frac{\cot \varphi \Theta^2}{\rho} \right) \frac{\partial f}{\partial \Phi} = 0. \end{aligned} \quad (39)$$

We shall deal with an equatorial belt and as such the problem could be regarded as essentially two-dimensional if we idealize the belt to

being a curve (as we shall do). However, we shall keep the three-dimensional equations and let the delta functions take care of this specialization automatically.

Assume at time $t = 0$ the needles are expelled from their container at a distance ρ_0 from the center of the earth in the longitudinal direction with a velocity dispersion function, in this direction only, given by $g(\Theta)$. That is to say, the initial velocity distribution of the particles imparted to them by the carrier rocket and the expulsion mechanism in the θ direction is denoted by $g(\Theta)$. This function will be taken as normalized so that its integral from $-\infty$ to $+\infty$ is equal to one, and it will also be taken to be symmetric about the circular orbit velocity $\rho_0 \omega$. Thus, we are to treat Liouville's Equation (39) as an initial value problem with the value of f at time $t = 0$ being

$$f_0 \equiv f(\rho, \theta, \varphi, P, \Theta, \Phi, 0) = \frac{M}{A^3} \delta(\rho - \rho_0) \delta(\theta) \delta(\varphi - \frac{\pi}{2}) \delta(P) \delta(\Phi) g(\Theta). \quad (40)$$

This says that the needles of total mass M are all concentrated initially at the point $(\rho_0, 0, \frac{\pi}{2})$ with zero velocity components in the ρ and φ directions but with a velocity distribution $g(\Theta)$ in the θ direction. The ρ_0^3 is present in the denominator because of the fact that the delta function in curvilinear coordinates when written as the direct product of delta functions of the individual coordinates must have the Jacobian of the transformation in the denominator ((3, p. 292)); for spherical coordinates the Jacobian is $\rho^2 \sin \varphi$ which gets converted into ρ_0^2 because of the presence of $\delta(\rho - \rho_0)$ and $\delta(\varphi - \frac{\pi}{2})$.

The method we adopt first for this initial value problem is that provided by the Picard process on t which essentially amounts to obtaining f as a power series in t . This has the advantage of being rather easily carried out but it has the disadvantage of presenting f in a form that does not display all the properties of the distribution function as clearly as in the case of certain other methods,--in particular, the method we illustrate after this one.

The idea is simple enough: it is to write (39) in the form

$$f = f_0 - \int_0^t \left[P \frac{\partial f}{\partial \rho} + \frac{\Theta}{\rho \sin \varphi} \frac{\partial f}{\partial \theta} + \frac{\Phi}{\rho} \frac{\partial f}{\partial \varphi} + \left(-\frac{\partial U}{\partial \rho} + \frac{\Theta^2}{\rho} + \frac{\Phi^2}{\rho} \right) \frac{\partial f}{\partial P} \right. \\ \left. + \left(-\frac{1}{\rho \sin \varphi} \frac{\partial U}{\partial \theta} - \frac{P\Theta}{\rho} - \frac{\omega t \varphi \Phi \Theta}{\rho} \right) \frac{\partial f}{\partial \Theta} + \left(-\frac{1}{\rho} \frac{\partial U}{\partial \varphi} - \frac{P\Phi}{\rho} + \frac{\omega t \varphi \Theta^2}{\rho} \right) \frac{\partial f}{\partial \Phi} \right] dt, \quad (41)$$

and employ iteration, starting with f replaced by f_0 in the integrand.

We consider the force field due to three sources: the earth's gravitational field, the self-gravitational field of the particles themselves, and the gravitational field of the needles' original container which presumably will be traveling right along with them after they have been ejected. The potential per unit mass due to the earth's pull is

$$U^{(1)} = - \frac{GM_E}{\rho}.$$

That due to the needles themselves is

$$U^{(2)} = -G \int_0^\pi \int_0^{2\pi} \int_0^\infty \frac{N(\bar{s}, \bar{\theta}, \bar{\varphi}, t) \bar{s}^2 \sin \bar{\varphi} d\bar{s} d\bar{\theta} d\bar{\varphi}}{\sqrt{\bar{s}^2 + \rho^2 - 2\rho\bar{s} [\cos \bar{\varphi} \cos \varphi + \sin \bar{\varphi} \sin \varphi \cos(\bar{\theta} - \theta)]}}$$

where $N(\rho, \theta, \varphi, t)$ is the spatial distribution of the needles. On the assumption that the container is traveling in a circular orbit in the midst of the needles we have for its potential per unit mass (M_c the mass of the container)

$$U^{(3)} = - \frac{G M_c}{\sqrt{\rho_0^2 + \rho^2 - 2\rho_0 \rho \sin \varphi \cos(\theta - \omega t)}}.$$

For the initial approximation we take $f = f_0$, so that

$$\begin{aligned} N_0 \equiv N(f_0) &= \iiint_{-\infty}^{\infty} f_0 dP d\Theta d\Phi, \\ &= \iiint_{-\infty}^{\infty} \frac{M}{\rho_0^2} \delta(\rho - \rho_0) \delta(\theta) \delta(\varphi - \frac{\pi}{2}) \delta(P) \delta(\Phi) g(\Theta) dP d\Theta d\Phi, \\ &= \frac{M}{\rho_0^2} \delta(\rho - \rho_0) \delta(\theta) \delta(\varphi - \frac{\pi}{2}). \end{aligned}$$

The initial approximation in the self-gravitation potential consequently is

$$\begin{aligned} U_0^{(3)} &= -G \int_0^\pi \int_0^{2\pi} \int_0^\infty \frac{N_0(\bar{\rho}, \bar{\theta}, \bar{\varphi}) \bar{\rho}^2 \sin \bar{\varphi} d\bar{\rho} d\bar{\theta} d\bar{\varphi}}{\sqrt{\bar{\rho}^2 + \rho^2 - 2\rho \bar{\rho} [\cos \bar{\varphi} \cos \varphi + \sin \bar{\varphi} \sin \varphi \cos(\bar{\theta} - \theta)]}}, \\ &= - \frac{G M}{\sqrt{\rho_0^2 + \rho^2 - 2\rho_0 \rho \sin \varphi \cos \theta}}. \end{aligned}$$

The initial potential due to the container is found by setting $t = 0$ in the formula for $U^{(3)}$:

$$U_o^{(3)} = - \frac{GM_c}{\sqrt{\rho_o^2 + \rho^2 - 2\rho_o\rho\sin\varphi\cos\theta}}.$$

Consequently, the combined initial potential is

$$U_o = - \frac{G(M+M_c)}{\sqrt{\rho_o^2 + \rho^2 - 2\rho_o\rho\sin\varphi\cos\theta}} - \frac{GM_E}{\rho}.$$

The terms involving the potential that arise in (41), $-\frac{\partial U_o}{\partial \rho} \frac{\partial f_o}{\partial P}$, $-\frac{1}{\rho\sin\varphi} \frac{\partial U_o}{\partial \theta} \frac{\partial f_o}{\partial \Theta}$, $-\frac{1}{\rho} \frac{\partial U_o}{\partial \varphi} \frac{\partial f_o}{\partial \Phi}$ simplify considerably due to the presence of the delta functions in f_o . In fact, we find

$$\frac{\partial U_o}{\partial \rho} \frac{\partial f_o}{\partial P} = \frac{GM_E}{\rho_o^2} \cdot \frac{M}{\rho_o^2} \delta(\rho-\rho_o) \delta(\theta) \delta(\varphi-\frac{\pi}{2}) \delta'(P) \delta(\Phi) g(\Theta),$$

$$\frac{1}{\rho\sin\varphi} \frac{\partial U_o}{\partial \theta} \frac{\partial f_o}{\partial \Theta} = \frac{1}{\rho} \frac{\partial U_o}{\partial \varphi} \frac{\partial f_o}{\partial \Phi} = 0.$$

Substituting f_o for f in the integrand of (41) then leads to the first order in t approximation to f :

$$\begin{aligned} f_\omega = & \frac{M}{\rho_o^2} \delta(\rho-\rho_o) \delta(\theta) \delta(\varphi-\frac{\pi}{2}) \delta(P) \delta(\Phi) g(\Theta) \\ & - \left\{ \frac{M}{\rho_o^3} \delta(\rho-\rho_o) \delta'(\theta) \delta(\varphi-\frac{\pi}{2}) \delta(P) \delta(\Phi) g(\Theta) \Theta \right. \\ & \left. + \left(\frac{M\Theta^2}{\rho_o^3} - \frac{GMM_E}{\rho_o^4} \right) \delta(\rho-\rho_o) \delta(\theta) \delta(\varphi-\frac{\pi}{2}) \delta'(P) \delta(\Phi) g(\Theta) \right\} t. \end{aligned}$$

This phase space distribution can be integrated over the velocities to give the position space density function through the first power in t . Simplifications occur here from the fact that

$$\int_{-\infty}^{\infty} \delta'(p) dp = 0,$$

and that $g(\Theta)$ had been assumed normalized so that its integral is equal to one. This latter specialization also entered into the computation for N_0 . This first order position space density is

$$N_{(1)} \equiv N(f_{(1)}) = \frac{M}{\rho_0^2} \delta(\rho - \rho_0) \delta(\theta) \delta(\varphi - \frac{\pi}{2}) - \frac{M}{\rho_0^2} \delta(\rho - \rho_0) \delta'(\theta) \delta(\varphi - \frac{\pi}{2}) \rho_0 \omega t.$$

The factor $\rho_0 \omega$ arises as a result of the integration of $\Theta g(\Theta)$, making use of the assumption that $g(\Theta)$ is symmetric about the point $\rho_0 \omega$.

$$\begin{aligned} \int_{-\infty}^{\infty} \Theta g(\Theta) d\Theta &= \int_{-\infty}^{\infty} [(\Theta - \rho_0 \omega) + \rho_0 \omega] g(\Theta) d\Theta, \\ &= \rho_0 \omega \int_{-\infty}^{\infty} g(\Theta) d\Theta + \int_{-\infty}^{\infty} (\Theta - \rho_0 \omega) g(\Theta) d\Theta, \\ &= \rho_0 \omega. \end{aligned}$$

Observe that the last integral vanishes since its integrand is skew-symmetric about $\rho_0 \omega$, i.e. an odd function of $\Theta - \rho_0 \omega$.

To obtain $f_{(1)}$ this entire procedure is repeated, using $f_{(1)}$ in place of f in the integrand of (41). This means also, of course, that $U_{(1)} \equiv U(f_{(1)})$ is to be employed. For the needles' self-gravitating potential it is thus

a matter of forming the potential integral with density $N_{(u)}$. The evaluation of this integral depends on the formula, used before, $\alpha(x)\delta'(x) = \alpha(0)\delta'(x) - \alpha'(0)\delta(x)$. The first order potential arising from the containers' mass is obtained simply by expanding its potential function in powers of t and keeping only the first two terms. The complete first order potential turns out to be

$$U_{(u)} \equiv U(f_u) = -\frac{GM_E}{r} - \frac{G(M+M_c)}{\sqrt{r_0^2 + r^2 - 2r_0 r \sin \varphi \cos \theta}} - \frac{G(M+M_c) \omega r_0 \sin \varphi \sin \theta}{[r_0^2 + r^2 - 2r_0 r \sin \varphi \cos \theta]^{3/2}} t.$$

The details in determining $f_{(u)}$ are similar to those for $f_{(u)}$ although the work is somewhat greater because of the larger number of terms involved. We find that

$$\begin{aligned} f_u = & \frac{M}{r_0^3} \delta(r-r_0) \delta(\theta) \delta(\varphi-\frac{\pi}{2}) \delta(P) \delta(\Phi) g(\Theta) \\ & - \left\{ \frac{M}{r_0^3} \delta(r-r_0) \delta'(\theta) \delta(\varphi-\frac{\pi}{2}) \delta(P) \delta(\Phi) g(\Theta) \Theta \right. \\ & \quad \left. + \frac{M}{r_0^3} \left(\frac{\Theta^2}{r_0^2} - \frac{GM_E}{r_0^3} \right) \delta(r-r_0) \delta(\theta) \delta(\varphi-\frac{\pi}{2}) \delta'(P) \delta(\Phi) g(\Theta) \right\} t \\ & - \frac{1}{2} \left\{ \frac{M}{r_0^3} \left(\frac{\Theta^2}{r_0^2} - \frac{GM_E}{r_0^3} \right) \delta'(r-r_0) \delta(\theta) \delta(\varphi-\frac{\pi}{2}) \delta(P) \delta(\Phi) g(\Theta) \right. \\ & \quad - \frac{M\Theta^2}{r_0^3} \delta(r-r_0) \delta''(\theta) \delta(\varphi-\frac{\pi}{2}) \delta(P) \delta(\Phi) g(\Theta) \\ & \quad - \frac{2M\Theta}{r_0^3} \left(\frac{\Theta^2}{r_0^2} - \frac{GM_E}{r_0^3} \right) \delta(r-r_0) \delta'(\theta) \delta(\varphi-\frac{\pi}{2}) \delta'(P) \delta(\Phi) g(\Theta) \\ & \quad - \frac{M}{r_0^3} \left(\frac{\Theta^2}{r_0^2} - \frac{GM_E}{r_0^3} \right)^2 \delta(r-r_0) \delta(\theta) \delta(\varphi-\frac{\pi}{2}) \delta''(P) \delta(\Phi) g(\Theta) \\ & \quad - \frac{G(M+M_c)M}{r_0^2 [r_0^2 + r^2 - 2r_0 r \sin \varphi \cos \theta]^{3/2}} \delta(r-r_0) \delta(\theta) \delta(\varphi-\frac{\pi}{2}) \delta(P) \delta(\Phi) [g(\Theta) + (\Theta-r_0)g'] \\ & \quad - \frac{M}{r_0^3} \left(\frac{\Theta^2}{r_0^2} + \frac{GM_E}{r_0^3} \right) \delta(r-r_0) \delta(\theta) \delta(\varphi-\frac{\pi}{2}) \delta(P) \delta(\Phi) g(\Theta) \\ & \quad \left. - \frac{M\Theta}{r_0^3} \left(\frac{\Theta^2}{r_0^2} - \frac{GM_E}{r_0^3} \right) \delta(r-r_0) \delta(\theta) \delta(\varphi-\frac{\pi}{2}) \delta(P) \delta(\Phi) g'(\Theta) \right\} t^2. \end{aligned} \quad (42)$$

There remains, of course, the essential problem of extracting the information of interest from this prolix expression. The principal difficulty arises in the meaning of the terms involving the derivatives of the delta functions. The most natural way of turning these terms to account is to take the various moments of f . These moments give average values for $\rho, \rho^2, \theta, \theta^2$, etc., for the collection of particles and thereby describe the distribution through these. Different delta derivatives take effect for different moments.

Of primary interest is the position space density function $N_{(u)} \equiv N(f_{(u)})$ generated by integrating $f_{(u)}$ over the velocity space. This is immediately accomplished from (42) and found to be

$$\begin{aligned} N_{(u)} = & \frac{M}{\rho_0^2} \delta(\rho-\rho_0) \delta(\theta) \delta(\varphi-\frac{\pi}{2}) - \frac{M\omega}{\rho_0^2} \delta(\rho-\rho_0) \delta'(\theta) \delta(\varphi-\frac{\pi}{2}) t \\ & - \frac{1}{2} \left\{ \frac{M}{\rho_0^2} \left(\int_{-\infty}^{\infty} \frac{u^2}{\rho_0} g(u) du - \frac{4M\epsilon}{\rho_0^2} \right) \delta'(\rho-\rho_0) \delta(\theta) \delta(\varphi-\frac{\pi}{2}) \right. \\ & \quad \left. - \frac{M}{\rho_0^4} \int_{-\infty}^{\infty} u^2 g(u) du \delta(\rho-\rho_0) \delta''(\theta) \delta(\varphi-\frac{\pi}{2}) \right. \\ & \quad \left. + \frac{2M}{\rho_0^3} \left(\int_{-\infty}^{\infty} \frac{u^2}{\rho_0} g(u) du - \frac{4M\epsilon}{\rho_0^2} \right) \delta(\rho-\rho_0) \delta(\theta) \delta(\varphi-\frac{\pi}{2}) \right\} t^2 \end{aligned} \quad (43)$$

The assumption was used that $(\theta-\rho_0\omega)g(\theta)$ vanishes at $\pm\infty$ so that the term involving $g(\theta) + (\theta-\rho_0\omega)g'(\theta) = \frac{d}{d\theta}[(\theta-\rho_0\omega)g(\theta)]$ vanishes under θ integration.

The integration of $N_{(u)}$ over all position space should, and does, give the total mass of the needles:

$$\int_0^\pi \int_0^{2\pi} \int_0^\infty N_{(u)} \rho^2 \sin \varphi d\rho d\theta d\varphi = M.$$

This provides one check on our work. The masses of the earth and of the container do not show up here since these bodies were not considered as included in the distribution function. They merely contribute the external force field.

At the initial instant the needles were all assumed to be concentrated at one point, but the continuous velocity distribution $g(\Theta)$ imposed on them then would cause them to spread. To get an idea to what extent this spreading has occurred (in the second order theory) in the radial direction we compute the ρ -moment of $N_{(1)}$. This, divided by the mass M , gives the average or expected value of ρ at time t .

$$\begin{aligned} M \overline{(\rho - \rho_0)} &\equiv \int_0^\pi \int_0^{2\pi} \int_0^\infty (\rho - \rho_0) N_{(1)} \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi, \\ &= -\frac{1}{2} \frac{M}{\rho_0^2} \left(\frac{1}{\rho_0} \int_{-\infty}^\infty u^2 g(u) du - \frac{GM_E}{\rho_0^2} \right) \int_0^\infty \rho^2 (\rho - \rho_0) \delta'(\rho - \rho_0) d\rho \cdot t^2, \\ &= \frac{1}{2} \frac{M}{\rho_0} I t^2, \end{aligned}$$

where we have indicated the second moment of $g(\Theta)$ about its point of symmetry, ρ_0 , by I :

$$\begin{aligned} I &\equiv \int_{-\infty}^\infty (u - \rho_0)^2 g(u) du, \\ &= \int_{-\infty}^\infty u^2 g(u) du - \rho_0^2 \bar{\omega}^2, \\ &= \int_{-\infty}^\infty u^2 g(u) du - \frac{GM_E}{\rho_0}, \end{aligned}$$

since for a circular orbit the angular velocity ω is $\sqrt{GM_E} \rho_0^{-3/2}$.
Therefore, the average or expected value of $\rho - \rho_0$ at time t is

$$\overline{(\rho - \rho_0)} = \frac{I}{2\rho_0} t^2$$

which we note is of the second order in t . Observe that the term in $N_{(2)}$ that did all the contributing to $M \overline{(\rho - \rho_0)}$ was the $\delta'(\rho - \rho_0)$ term, as all the others vanished under the integration.

The spread in the θ direction of the needles after expulsion from the container is indicated by the θ moments of $N_{(2)}$. We quickly find that

$$M \bar{\theta} = \frac{M \rho_0 \omega}{\rho_0^2} \rho_0^2 t = M \omega t$$

so that the average value of θ is, as to be expected from the symmetry of the setup,

$$\bar{\theta} = \omega t.$$

This, of course, tells us nothing about the spread of the needles in the θ direction. An idea of this is gained from the second moment of $N_{(2)}$ with respect to θ , i.e. (when divided by M) the average value of θ^2 . The only term that contributes to this in $N_{(2)}$ is the $\delta''(\theta)$ term, since $\theta^2 \delta(\theta) = \theta^2 \delta'(\theta) = 0$ and $\theta^2 \delta''(\theta) = 2 \delta(\theta)$. We obtain

$$M \bar{\theta^2} = \frac{M}{\rho_0^2} (I + \rho_0^2 \omega^2) t^2,$$

whence

$$\sqrt{\bar{\theta^2}} = \omega t \sqrt{1 + \frac{1}{\rho_0^2 \omega^2} I}.$$

Naturally, one can also compute the moments in the velocity space and thereby gain information concerning the average values of the different velocity components.

Our chief dissatisfaction with f or N obtained by the power series in t method is that it does not give a compact formula for these independent of the delta function where possible. For example, the density distribution in the θ direction at time t is no longer degenerate but is continuous, i.e. the needles have been spread out, and so there should be a function representing this that involves no delta functions. Such a function was not obtained by the method just used, but we proceed to illustrate a method that does provide this form for the answer.

A perusal of (42) indicates that as far as it goes f involves φ and Φ only through the factors $\delta(\varphi - \Xi)$ and $\delta(\Phi)$ (which has the effect of keeping the problem two-dimensional), and P only through $\delta(P)$ and its derivatives. This suggests then that we assume f to be of the form

$$f = \sum_{n=0}^{\infty} E_n(\rho, \theta, \Theta, t) \delta^{(n)}(P) \delta(\Phi) \delta(\varphi - \Xi). \quad (44)$$

We shall restrict our treatment to cover only the case of the force field consisting only of that resulting from the earth's mass. Consequently, we are considering the equation resulting from specializing (39) accordingly,

$$\begin{aligned} \frac{\partial f}{\partial t} + P \frac{\partial f}{\partial \rho} + \frac{\Theta}{\rho \sin \varphi} \frac{\partial f}{\partial \theta} + \frac{\Phi}{\rho} \frac{\partial f}{\partial \varphi} + \left(-\frac{GM_E}{\rho^2} + \frac{\Theta^2}{\rho} + \frac{\Phi^2}{\rho} \right) \frac{\partial f}{\partial P} \\ + \left(-\frac{P\Theta}{\rho} - \frac{\omega \varphi \Phi \Theta}{\rho} \right) \frac{\partial f}{\partial \Theta} + \left(-\frac{P\Phi}{\rho} + \frac{\omega \varphi \Theta^2}{\rho} \right) \frac{\partial f}{\partial \Phi} = 0. \end{aligned} \quad (45)$$

The initial condition is again presented by (40) which implies for the unknown functions $E_n(\rho, \theta, \Theta, t)$ that

$$\begin{cases} E_0(\rho, \theta, \Theta, 0) = \frac{M}{\rho_0^2} \delta(\rho - \rho_0) \delta(\theta) g(\Theta), \\ E_n(\rho, \theta, \Theta, 0) = 0, \quad n > 0. \end{cases} \quad (46)$$

The differential-difference equation is obtained for $E_n(\rho, \theta, \Theta, t)$ by substituting (44) in (45), sliding summation indices so that all sums contain $\delta^{(n)}(P) \delta(\Phi) \delta(\varphi - \frac{t}{\rho})$ as a factor, combining into one big sum and setting the coefficient of $\delta^{(n)}(P) \delta(\Phi) \delta(\varphi - \frac{t}{\rho})$ to zero. This coefficient has to vanish in order to make the series vanish since the coefficient is independent of P, Φ, φ . This fundamental equation for $E_n(\rho, \theta, \Theta, t)$ is

$$\frac{\partial E_n}{\partial t} - (n+1) \frac{\partial E_{n+1}}{\partial \rho} + \frac{\Theta}{\rho} \frac{\partial E_n}{\partial \theta} + \left(\frac{\Theta^2}{\rho} - \frac{GM_F}{\rho^2} \right) E_{n-1} + (n+1) \frac{\Theta}{\rho} \frac{\partial E_{n+1}}{\partial \Theta} - (n+1) \frac{1}{\rho} E_{n+1} = 0 \quad (47)$$

We are to take $E_r = 0$ for $r < 0$.

This equation can be regarded as a recursion formula in conjunction with an iteration procedure. For first approximation we take $E_1 = E_2 = E_3 = \dots = 0$. Then it may be verified that the solution that satisfies (46) is

$$E_0^{(0)} = \frac{M}{\rho_0^2} \delta(\rho - \rho_0) g(\Theta) \delta\left(\theta - \frac{\Theta t}{\rho_0}\right) = \frac{M}{\rho_0^2} \delta(\rho - \rho_0) g(\Theta) \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\left(\theta - \frac{\Theta t}{\rho_0}\right)}, \quad (48)$$

where the series is the Fourier representation for the delta function if the argument of the delta function is so restricted that periodicity in the θ doesn't enter (8, p. 101). This representation is introduced here to facilitate the use of the Laplace Transformation in solving for $E_1^{(u)}$ and others.

To improve on this approximation to E_0 we first determine an approximate E_1 using the value of $E_0^{(u)}$ in the equation for E_1 (setting $E_2 = 0$) and then use this value of E_1 back in the equation for E_0 . The first three equations pulled from (47) are

$$\begin{cases} \frac{\partial E_0}{\partial t} + \frac{\Theta}{\rho} \frac{\partial E_0}{\partial \theta} = \frac{\partial E_1}{\partial \rho} - \frac{\Theta}{\rho} \frac{\partial E_1}{\partial \theta} + \frac{1}{\rho} E_1, \\ \frac{\partial E_1}{\partial t} + \frac{\Theta}{\rho} \frac{\partial E_1}{\partial \theta} = -\left(\frac{\Theta^2}{\rho^2} - \frac{GM_E}{\rho^2}\right)E_0 + 2\left(\frac{\partial E_2}{\partial \rho} - \frac{\Theta}{\rho} \frac{\partial E_2}{\partial \theta} + \frac{1}{\rho} E_2\right), \\ \frac{\partial E_2}{\partial t} + \frac{\Theta}{\rho} \frac{\partial E_2}{\partial \theta} = -\left(\frac{\Theta^2}{\rho^2} - \frac{GM_E}{\rho^2}\right)E_1 + 3\left(\frac{\partial E_3}{\partial \rho} - \frac{\Theta}{\rho} \frac{\partial E_3}{\partial \theta} + \frac{1}{\rho} E_3\right). \end{cases} \quad (49)$$

$E_0^{(u)}$ is the solution to the first of (49) (with side condition (46)) with E_1 set to zero. We thus seek $E_1^{(u)}$ as a solution to the second of (49) with E_2 set to zero, E_0 replaced by $E_0^{(u)}$, and the initial value of E_1 taken to be zero as (46) says. That is, we are to solve for zero initial value the equation

$$\frac{\partial E_1^{(u)}}{\partial t} + \frac{\Theta}{\rho} \frac{\partial E_1^{(u)}}{\partial \theta} = -\frac{M}{\rho_0^2} \delta(\rho - \rho_0) g(\Theta) \left(\frac{\Theta^2}{\rho^2} - \frac{GM_E}{\rho^2}\right) \frac{1}{2\pi} \sum e^{in(\theta - \frac{\Theta t}{\rho_0})}. \quad (50)$$

The Laplace Transform of (50) with respect to t is

$$\tau \bar{E}_1^{(\omega)} + \frac{\Theta}{\rho_0^2} \frac{\partial \bar{E}_1^{(\omega)}}{\partial \theta} = -\frac{M}{\rho_0^2} \delta(\rho-\rho_0) g(\Theta) \left(\frac{\Theta^2}{\rho_0^2} - \frac{GM_E}{\rho_0^2} \right) \frac{1}{2\pi} \sum \frac{e^{in\theta}}{\tau + \frac{in\Theta}{\rho_0}}. \quad (51)$$

By assuming a Fourier expansion for $\bar{E}_1^{(\omega)}$,

$$\bar{E}_1^{(\omega)} = \sum_{-\infty}^{\infty} \bar{c}_n e^{in\theta}$$

we obtain from (51) that

$$\bar{E}_1^{(\omega)} = -\frac{M}{\rho_0^2} \delta(\rho-\rho_0) g(\Theta) \left(\frac{\Theta^2}{\rho_0^2} - \frac{GM_E}{\rho_0^2} \right) \frac{1}{2\pi} \sum \frac{e^{in\theta}}{(\tau + \frac{in\Theta}{\rho_0})^2},$$

from which we conclude upon taking the inverse transform that

$$\begin{aligned} E_1^{(\omega)} &= -\frac{M}{\rho_0^2} \delta(\rho-\rho_0) g(\Theta) \left(\frac{\Theta^2}{\rho_0^2} - \frac{GM_E}{\rho_0^2} \right) \frac{1}{2\pi} \sum t e^{in(\theta - \frac{\Theta t}{\rho_0})}, \\ &= -\frac{M}{\rho_0^2} \delta(\rho-\rho_0) g(\Theta) \left(\frac{\Theta^2}{\rho_0^2} - \frac{GM_E}{\rho_0^2} \right) t \delta(\theta - \frac{\Theta t}{\rho_0}). \end{aligned}$$

We use this same technique to determine $E_0^{(\omega)}$ from the first of (49) and discover that

$$\begin{aligned} E_0^{(\omega)} &= \frac{M}{\rho_0^2} \delta(\rho-\rho_0) g(\Theta) \delta(\theta - \frac{\Theta t}{\rho_0}) \\ &\quad + \left[-\frac{M}{\rho_0^2} \delta'(\rho-\rho_0) g(\Theta) \left(\frac{\Theta^2}{\rho_0^2} - \frac{GM_E}{\rho_0^2} \right) + \frac{M\Theta}{\rho_0^2} \delta(\rho-\rho_0) g'(\Theta) \left(\frac{\Theta^2}{\rho_0^2} - \frac{GM_E}{\rho_0^2} \right) \right. \\ &\quad \left. + \frac{M}{\rho_0^2} \delta(\rho-\rho_0) g(\Theta) \left(\frac{\Theta^2}{\rho_0^2} + \frac{GM_E}{\rho_0^2} \right) \right] \frac{1}{2} t^2 \delta(\theta - \frac{\Theta t}{\rho_0}) \\ &\quad - \frac{M\Theta}{\rho_0^2} \delta(\rho-\rho_0) g(\Theta) \left(\frac{\Theta^2}{\rho_0^2} - \frac{GM_E}{\rho_0^2} \right) \frac{1}{3} t^3 \delta'(\theta - \frac{\Theta t}{\rho_0}). \end{aligned} \quad (52)$$

This successive approximation scheme for the system (49) is in essence the same as that suggested for (22) in the section devoted to the improvement of an approximate orbit of a single particle or n particles. In that theory it was c_{λ_i, s_i} with $\lambda_1 + \dots + s_3 = 1$ that was of greatest interest. In the present case it is E_0 because it is the only E_r that remains upon reduction to position space by integrating over the velocities. The expression for N , the position space density, is simply

$$N(f) = \iiint_{-\infty}^{\infty} f dP d\Theta d\Phi = \int_{-\infty}^{\infty} E_0(\rho, \theta, \Theta, t) d\Theta \delta(\varphi - \frac{\pi}{2}).$$

When the value for E_0 given by (52) is employed the spatial density is found to be

$$\begin{aligned} N_{\omega} \equiv N(f^{(\omega)}) &= \frac{M}{\rho_0^2} \delta(\rho - \rho_0) \frac{\rho_0}{t} g\left(\frac{\rho_0 \theta}{t}\right) \\ &+ \frac{M}{\rho_0^2} \delta(\varphi - \frac{\pi}{2}) \left\{ -\frac{\rho_0^2}{2} \left(\frac{\rho_0 \theta^2}{t^2} - \frac{GM_E}{\rho_0^2} \right) \delta'(\rho - \rho_0) g\left(\frac{\rho_0 \theta}{t}\right) \right. \\ &\quad \left. + \left(-\frac{\rho_0 \theta^2}{2t} + \frac{5GM_E t}{6\rho_0^2} \right) \delta(\rho - \rho_0) g\left(\frac{\rho_0 \theta}{t}\right) + \left(\frac{\rho_0^3 \theta^3}{6t^2} - \frac{GM_E \theta}{6\rho_0} \right) \delta(\rho - \rho_0) g'\left(\frac{\rho_0 \theta}{t}\right) \right\}. \end{aligned} \quad (53)$$

The interesting thing about this formula in comparison with the corresponding formula (43) of the previous method is that in (53) the θ, t dependence is expressed through the continuous point function $g(u)$ instead of through the delta function and its derivatives. Note that this method has transferred the continuous initial velocity distribution in Θ over to such a distribution in the position space coordinate θ .

One can, of course, form the various moments with respect to the $N_{(2)}$ given in (53) as in the previous method and obtain the same results for

$\overline{\rho-\rho_0}$, $\overline{\theta}$, $\overline{\theta^2}$, etc., as in the previous method. However, the particular form for (53) enables us to get other information of interest. For example, the longitudinal variation of the needle density is nicely displayed by the formula resulting from integrating $N_{(2)}$ in the ρ and φ directions:

$$\int_0^\infty \int_0^\pi N_{(2)} \rho d\varphi d\rho = \frac{M}{\rho_0} \left\{ \left[1 + \frac{1}{3} \omega^2 t^2 \right] \frac{A_0^2}{t^2} g\left(\frac{A_0 \theta}{t}\right) + \frac{1}{6} \theta (\theta^2 \omega^2 t^2) \frac{A_0^2}{t^2} g'\left(\frac{A_0 \theta}{t}\right) \right\}.$$

Including the self-gravitation effects of the needles adds to the difficulty of the problem inasmuch as (47) is then non-linear since E_0 will occur in the potential integral forming part of the coefficient to E_{n-1} . It can, of course, be linearized by using some given approximate value for E_0 in the integrand with the result later improved on by iteration.

Suppose the dispensing mechanism has accomplished its mission and has spread the needles out in a belt around the earth. The question then arises as to the permanence of the belt under the action of the various disturbing forces present.

We idealize the unperturbed distribution to have the shape of a circular wire of uniform density in the equatorial plane. The particles are then to have non-zero velocity component only in the longitudinal direction and of such magnitude as to give a circular orbit under the earth's gravitational field alone. The distribution function that describes this is

$$\hat{f} = \frac{M}{2\pi a^2} \delta(\rho-\rho_0) \delta(\varphi-\frac{\pi}{2}) \delta(P) \delta(\Theta-\rho_0 \omega) \delta(\Phi). \quad (54)$$

The 2π in the denominator is necessary so that the integral of $\hat{f}$ over all phase space comes out to be simply M , the total mass of the needles.

This $\hat{f}$ represents an equilibrium distribution under the earth's gravitational force alone and so it is a steady state solution to Liouville's Equation. At time $t = 0$ assume the perturbational forces, represented through a potential U , to take effect, causing the distribution function to become modified by the addition of a function $\hat{F}$ which vanishes at $t = 0$:

$$f = \hat{f} + \hat{F}.$$

This is equivalent to saying that f is to satisfy an initial value problem under the combined earth's gravitational and perturbational force fields with initial value $\hat{f}$ as in (54).

Under these circumstances Liouville's Equation becomes an equation for $\hat{F}$,

$$\begin{aligned} & \frac{\partial \hat{F}}{\partial t} + P \frac{\partial \hat{F}}{\partial p} + \frac{\Theta}{r \sin \varphi} \frac{\partial \hat{F}}{\partial \theta} + \frac{\Phi}{r} \frac{\partial \hat{F}}{\partial \varphi} + \left(-\frac{\partial U}{\partial r} - \frac{GM_E}{r^2} + \frac{\Theta^2}{r} + \frac{\Phi^2}{r} \right) \frac{\partial \hat{F}}{\partial P} \\ & - \frac{\partial U}{\partial p} \frac{M}{2\pi r_0^2} \delta(r-r_0) \delta(\varphi-\frac{\pi}{2}) \delta'(P) \delta(\Theta-r_0 \omega) \delta(\Phi) + \left(-\frac{1}{r \sin \varphi} \frac{\partial U}{\partial \theta} - \frac{P\Theta}{r} - \frac{\omega \varphi \Phi \Theta}{r} \right) \frac{\partial \hat{F}}{\partial \Theta} \\ & - \frac{1}{r \sin \varphi} \frac{\partial U}{\partial \varphi} \frac{M}{2\pi r_0^2} \delta(r-r_0) \delta(\varphi-\frac{\pi}{2}) \delta(P) \delta'(\Theta-r_0 \omega) \delta(\Phi) + \left(-\frac{1}{r} \frac{\partial U}{\partial \varphi} - \frac{P\Phi}{r} + \frac{\omega \varphi \Theta^2}{r} \right) \frac{\partial \hat{F}}{\partial \Phi} \\ & - \frac{1}{r} \frac{\partial U}{\partial \varphi} \frac{M}{2\pi r_0^2} \delta(r-r_0) \delta(\varphi-\frac{\pi}{2}) \delta(P) \delta(\Theta-r_0 \omega) \delta'(\Phi) = 0. \end{aligned} \quad (55)$$

In order that the problem stay two-dimensional for simplicity's sake, the disturbing force in the equatorial plane must act only parallel to this plane. A way of expressing this is to demand that $\delta(\varphi-\frac{\pi}{2}) \frac{\partial U}{\partial \varphi} = 0$, which we shall do.

Let

$$\hat{F} = \delta(\varphi-\frac{\pi}{2}) \delta(\Phi) F(r, \theta, P, \Theta, t)$$

and substitute in (55) to obtain

$$\begin{aligned} \frac{\partial F}{\partial t} + P \frac{\partial F}{\partial \rho} + \frac{\Theta}{\rho} \frac{\partial F}{\partial \theta} + \left(-\frac{\partial \bar{U}}{\partial \rho} - \frac{GM_E}{\rho^2} + \frac{\Theta^2}{\rho} \right) \frac{\partial F}{\partial P} - \frac{\partial \bar{U}}{\partial \rho} \frac{M}{2\pi \rho^2} \delta(\rho - \rho_0) \delta'(P) \delta(\Theta - \rho_0 \omega) \\ + \left(-\frac{1}{\rho} \frac{\partial \bar{U}}{\partial \theta} - \frac{P\Theta}{\rho} \right) \frac{\partial F}{\partial \Theta} - \frac{1}{\rho_0} \frac{\partial \bar{U}}{\partial \theta} \frac{M}{2\pi \rho_0^2} \delta(\rho - \rho_0) \delta(P) \delta'(\Theta - \rho_0 \omega) + \frac{1}{\rho} P F = 0, \end{aligned} \quad (56)$$

where $\bar{U}$ is U evaluated for $\varphi = \frac{\pi}{2}$. This equation is to be solved for F under the condition that F vanish at $t = 0$. Since we are primarily interested in the character of F in position space we find it expedient to assume an expansion for F of the form

$$F(\rho, \theta, P, \Theta, t) = \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} u_{ns}(\rho, \theta, t) \delta^{(n)}(P) \delta^{(s)}(\Theta - \rho_0 \omega). \quad (57)$$

Thus, the delta functions in (57) get integrated out in finding the position space density, so that N becomes expressed in terms of $u_{ns}(\rho, \theta, t)$. We hope to be able to determine $u_{ns}(\rho, \theta, t)$ in a form devoid of delta functions of θ .

The equation in u_{ns} resulting from combining (57) and (56) is, after some delta function manipulations,

$$\begin{aligned} \sum_{n,s} \left[\frac{\partial u_{ns}}{\partial t} - (n+1) \frac{\partial u_{n+1,s}}{\partial \rho} + \frac{\rho_0 \omega}{\rho} \frac{\partial u_{ns}}{\partial \theta} - \frac{s+1}{\rho} \frac{\partial u_{n,s+1}}{\partial \theta} - \left(\frac{\partial \bar{U}}{\partial \rho} + \frac{GM_E}{\rho^2} \right) u_{n-1,s} + \frac{\rho_0^2 \omega^2}{\rho} u_{n-1,s} - \frac{2(n+1)\rho_0 \omega}{\rho} u_{n-1,s+1} \right. \\ \left. + 2 \binom{s+2}{2} \frac{1}{\rho} u_{n-1,s+2} - \frac{1}{\rho} \frac{\partial \bar{U}}{\partial \theta} u_{n,s-1} + \frac{(n+1)\rho_0 \omega}{\rho} u_{n+1,s-1} - \frac{(n+1)(s+1)}{\rho} u_{n+1,s} - \frac{1}{\rho} u_{n+1,s} \right] \delta^{(n)}(P) \delta^{(s)}(\Theta - \rho_0 \omega) \\ - \frac{\partial \bar{U}}{\partial \rho} \frac{M}{2\pi \rho_0^2} \delta(\rho - \rho_0) \delta'(P) \delta(\Theta - \rho_0 \omega) - \frac{1}{\rho_0} \frac{\partial \bar{U}}{\partial \theta} \frac{M}{2\pi \rho_0^2} \delta(\rho - \rho_0) \delta(P) \delta'(\Theta - \rho_0 \omega) = 0. \end{aligned} \quad (58)$$

We shall consider two types of disturbing forces: solar radiation pressure, and drag due to electromagnetic effects (10, 11). The solar pressure will be taken as a constant force acting in the negative x direction with earth's shadow effect neglected. The potential then has the form $S\rho\cos\theta\sin\varphi$, with S having dimensions LT^{-2} . The electromagnetic drag can be approximated to sufficient accuracy for particles traversing a circular or near circular orbit by a conservative force with potential $A\theta$, with A having dimensions L^2T^{-2} . Consequently, the disturbing potential is taken to be

$$U = A\theta + S\rho\cos\theta\sin\varphi. \quad (59)$$

The functions u_{hs} are to satisfy the system of equations derived from (58) using (59)

$$\left\{ \begin{aligned} \frac{\partial u_{00}}{\partial t} + \frac{\rho_0 \omega}{\rho} \frac{\partial u_{00}}{\partial \theta} &= \frac{1}{\rho} \frac{\partial u_{01}}{\partial \theta} + \frac{\partial u_{10}}{\partial \rho} + \frac{2}{\rho} u_{10}, \\ \frac{\partial u_{01}}{\partial t} + \frac{\rho_0 \omega}{\rho} \frac{\partial u_{01}}{\partial \theta} + \frac{\rho_0 \omega}{\rho} u_{10} &= \frac{1}{\rho} (A - S\rho\sin\theta) u_{00} + \frac{2}{\rho} \frac{\partial u_{02}}{\partial \theta} + \frac{\partial u_{11}}{\partial \rho} + \frac{3}{\rho} u_{11} \\ &\quad + \frac{1}{\rho_0} (A - S\rho_0\sin\theta) \frac{M}{2\pi\rho_0^2} \delta(\rho - \rho_0), \\ \frac{\partial u_{10}}{\partial t} + \frac{\rho_0 \omega}{\rho} \frac{\partial u_{10}}{\partial \theta} - 2 \frac{\rho_0 \omega}{\rho} u_{01} &= (S\cos\theta + \frac{GM_E}{\rho^2} - \frac{\rho_0^2 \omega^2}{\rho}) u_{00} - \frac{2}{\rho} u_{02} + \frac{1}{\rho} \frac{\partial u_{11}}{\partial \theta} \\ &\quad + 2 \frac{\partial u_{20}}{\partial \rho} + \frac{3}{\rho} u_{20} + S\cos\theta \frac{M}{2\pi\rho_0^2} \delta(\rho - \rho_0), \\ &\dots \end{aligned} \right. \quad (60)$$

under the initial condition of all the u_{hs} vanishing at the initial instant.

This problem submits to the same technique used earlier on similar problems. To obtain the first approximation to u_{00} solve the first of (60) with u_{01} and u_{10} set to zero. This homogeneous equation with zero initial data evidently has as solution $u_{00}^{(1)} = 0$. For first approximation to u_{01} and u_{10} put $u_{00} = u_{00}^{(0)} = 0$ and set $u_{02} = u_{11} = u_{20} = 0$ in the second and third equations in (60):

$$\begin{cases} \frac{\partial u_{01}^{(1)}}{\partial t} + \frac{\rho_0 \omega}{\rho} \frac{\partial u_{01}^{(1)}}{\partial \theta} + \frac{\rho_0 \omega}{\rho} u_{10}^{(1)} = \frac{1}{\rho_0} (A - S \rho_0 \sin \theta) \frac{M}{2\pi \rho_0^2} \delta(\rho - \rho_0), \\ \frac{\partial u_{10}^{(1)}}{\partial t} + \frac{\rho_0 \omega}{\rho} \frac{\partial u_{10}^{(1)}}{\partial \theta} - 2 \frac{\rho_0 \omega}{\rho} u_{01}^{(1)} = S \cos \theta \frac{M}{2\pi \rho_0^2} \delta(\rho - \rho_0). \end{cases} \quad (61)$$

This system of linear partial differential equations with constant coefficients (as far as the differentiation variables are concerned) subject to the side conditions of zero initial data on t and periodicity on θ can be handled by various techniques. One convenient one is that used previously: take Laplace transform with respect to t , solve the resulting system of total differential equations in θ , and then take the inverse transform. The result in this case is

$$\begin{cases} u_{01}^{(1)} = \frac{AM \delta(\rho - \rho_0)}{2\pi \rho_0^3 \sqrt{2} \omega} \sin \sqrt{2} \omega t - \frac{MS \delta(\rho - \rho_0)}{2\pi \rho_0^3 \omega} \left[2 \cos \theta - \frac{4+3\sqrt{2}}{4} \cos(\theta + (\sqrt{2}-1)\omega t) - \frac{4-3\sqrt{2}}{4} \cos(\theta - (\sqrt{2}+1)\omega t) \right], \\ u_{10}^{(1)} = \frac{AM \delta(\rho - \rho_0)}{2\pi \rho_0^3 \omega} (1 - \cos \sqrt{2} \omega t) + \frac{MS \delta(\rho - \rho_0)}{2\pi \rho_0^3 \omega} \left[-3 \sin \theta + \frac{3+2\sqrt{2}}{2} \sin(\theta + (\sqrt{2}-1)\omega t) + \frac{3-2\sqrt{2}}{2} \sin(\theta - (\sqrt{2}+1)\omega t) \right]. \end{cases} \quad (62)$$

We therefore improve on the value of $u_{00}^{(1)}$ by solving the first equation in (60) after having evaluated the right side through the use of (62). The formula obtained for u_{00} this time is

$$u_{00}^{(2)} = \frac{AM}{2\pi\rho_0^3\omega} (\delta'(\rho-\rho_0) + \frac{2}{\rho_0}\delta(\rho-\rho_0))(t - \frac{1}{\sqrt{2}\omega}\sin\sqrt{2}\omega t) \\ + \frac{SM}{2\pi\rho_0^3\omega} \left\{ \left(\frac{3}{\omega}\delta'(\rho-\rho_0) + \frac{1}{\rho_0\omega}\delta(\rho-\rho_0) \right) \cos\theta - \frac{1}{\omega}\delta'(\rho-\rho_0)\cos(\theta-\omega t) - \frac{1}{\rho_0}\delta(\rho-\rho_0)\sin(\theta-\omega t) \right. \\ \left. - \left(\frac{4+3\sqrt{2}}{4\omega}\delta'(\rho-\rho_0) + \frac{1+\sqrt{2}}{2\rho_0\omega}\delta(\rho-\rho_0) \right) \cos(\theta + (\sqrt{2}-1)\omega t) \right. \\ \left. - \left(\frac{4-3\sqrt{2}}{4\omega}\delta'(\rho-\rho_0) + \frac{1-\sqrt{2}}{2\rho_0\omega}\delta(\rho-\rho_0) \right) \cos(\theta - (\sqrt{2}+1)\omega t) \right\}. \quad (63)$$

To obtain a still better approximation one would find $u_{01}^{(1)}, u_{11}^{(1)}, u_{20}^{(1)}$ from equations four, five and six in (60), use these values and $u_{00}^{(2)}$ in equations two and three to get $u_{01}^{(2)}, u_{10}^{(2)}$ and then finally put in these values in the first equation in (60) and solve for u_{00} .

An analysis of our results would begin with the determination of the position space density. It is

$$N = \iiint_{-\infty}^{\infty} (\hat{f} + \hat{F}) dP d\Theta d\Phi = \frac{M}{2\pi\rho_0^3} \delta(\rho-\rho_0) \delta(\varphi-\frac{\pi}{2}) + u_{00} \delta(\varphi-\frac{\pi}{2}).$$

The variation of the density with θ is indicated by the integration of N in the ρ and φ directions. The result is

$$\int_0^\infty \int_0^\pi N_\omega \rho d\varphi d\theta = \frac{M}{2\pi\rho_0} + \frac{AM}{2\pi\rho_0^3\omega} (t - \frac{1}{\sqrt{2}\omega}\sin\sqrt{2}\omega t) \\ + \frac{SM}{2\pi\rho_0^3\omega} \left[-\frac{2}{\omega}\cos\theta + \frac{1}{\omega}\cos(\theta-\omega t) - t\sin(\theta-\omega t) + \frac{2+\sqrt{2}}{4\omega}\cos(\theta + (\sqrt{2}-1)\omega t) + \frac{2-\sqrt{2}}{4\omega}\cos(\theta - (\sqrt{2}+1)\omega t) \right], \\ = \frac{M}{2\pi\rho_0} \left\{ 1 + \frac{A}{\rho_0^2\omega^2} \left(\frac{1}{3}\omega^3 t^3 + \dots \right) \right. \\ \left. + \frac{S}{\rho_0\omega^2} \left[(-2\omega^2 t^2 + \dots) \cos\theta + \left(\frac{1}{15}\omega^5 t^5 + \dots \right) \sin\theta \right] \right\}. \quad (64)$$

If one considers the radiation pressure effect alone he notes from (64) that the density runs essentially as $\frac{M}{2\pi\rho_0} [1 - k\omega\theta]$, meaning that the density concentration increases on the far side from the sun and decreases on the near side.

An insight into the radial direction effects of the disturbing forces is gained from a study of the radial moment. A straightforward computation gives us

$$\begin{aligned} \int_0^\infty \int_0^\pi (\rho - \rho_0) N_{(2)} \rho d\phi d\rho &= -\frac{AM}{2\pi\rho_0^2\omega} \left(t - \frac{1}{\sqrt{2}} \sin \sqrt{2}\omega t \right) \\ &\quad + \frac{SM}{2\pi\rho_0^2\omega} \left[-\frac{3\rho_0}{\omega} \cos \theta + \frac{\rho_0}{\omega} \cos(\theta - \omega t) + \frac{4+3\sqrt{2}}{4\omega} \rho_0 \cos(\theta + (\sqrt{2}-1)\omega t) \right. \\ &\quad \left. + \frac{4-3\sqrt{2}}{4\omega} \rho_0 \cos(\theta - (\sqrt{2}+1)\omega t) \right], \\ &= \frac{M}{2\pi} \left\{ -\frac{A}{\rho_0^2\omega^2} \left(\frac{1}{3}\omega^3 t^3 + \dots \right) + \frac{S}{\rho_0\omega^2} \left[\left(-\frac{1}{2}\omega^2 t^2 + \dots \right) \cos \theta + \left(-\frac{1}{30}\omega^5 t^5 + \dots \right) \sin \theta \right] \right\}. \end{aligned} \quad (65)$$

The average value of $\rho - \rho_0$ as a function of θ comes as a result of dividing (65) by (64). The electromagnetic drag is seen from this to contribute a uniform shrinkage to the belt with change in ρ amounting to $-\frac{A}{32\omega^2} (\omega t)^3 + \dots$ which for moderate t is probably a very small quantity due to the probable small size of A (11). The solar radiation pressure evidently causes a shift of the entire orbit in the direction away from the sun together with a much lesser shift in the 90° leading direction to the former. These we note from the negative lead-off coefficients of $\cos \theta$ and $\sin \theta$ in (65). The shape of the belt becomes altered, too,--from a circle to a near limaçon with the nearest point to the earth almost on the line from the earth to the sun.

The function $\hat{f}$ given in (54) does not satisfy Liouville's Equation if self-gravitation of the particles is taken into account. If the needles

somehow had gotten themselves arranged into the distribution $\hat{f}$, self-gravitation would act as a disturbing force and cause $\hat{f}$ to change. It is apparent that the effect would be to cause a uniform shrinkage. However, it is also apparent that this effect would be very small due to the small total mass of the needles which are themselves scattered all the way around the earth. It would not be difficult to determine $\mu_{00}^{(2)}$ under the effects of the self-gravitating potential but we will not carry through the details here.

Conclusion:

The foregoing examples make evident the usefulness and convenience of the delta functions in various types of problems in mechanics. The procedure has application both as a method for deriving the fundamental equations in a particular case and as a tool for presenting the equations often in a form somewhat different from the conventional so that new methods of attack may present themselves. To make it fully satisfactory from a mathematical point of view, work needs to be done to keep it all within the framework of the Schwartz distribution theory or a similar one. For example, extension of the definition of multiplication of weak functions to cover products of discontinuous point functions with delta functions (such as those briefly touched on on p. 9) should be investigated as well as existence of weak function solutions to the non-linear problems that arise.

Ray G. Langebartel
Ray G. Langebartel
Staff Member
Plans Office

F. O. Vonbun
F. O. Vonbun
Head, Plans Office

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